Fourth order ETDRK scheme with Cauchy integral for nonlinear dispersive wave equations

Muyassar Ahmat and Jianxian Qiu

Abstract

In this paper, a fourth-order scheme is presented for nonlinear dispersive wave equations. The scheme uses the fourth-order compact finite difference method for discretization in space and fourth-order exponential time differencing Runge-Kutta (ETDRK) method for the temporal direction, respectively. The Cauchy integral formula takes effect on stabilizing the fourth-order ETDRK method, and deals with non-diagonal large sparse coefficient matrix which has complex eigenvalues tend to zero. The Krylov subspace with the Arnoldi algorithm is used to reduce the computational cost on the exponential computation in the linear part. It can be observed by numerical experiments that the numerical method is performed efficiently for the solitary wave profile of the Rosenau-KDV-RLW equation.

Key Words: Nonlinear dispersive wave equation, Fourth-order compact scheme, Finite difference method, ETD Runge-Kutta method, Cauchy integral formula, Krylov subspace.

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1 Introduction

Many natural phenomena can be modeled by various nonlinear equations mathematically, and yet it is been challenging to evaluate analytical and numerical solutions of these types of nonlinear equations. The exact solutions of these nonlinear equations are always hard to be found, and even analytical solutions are barely available because the nonlinear terms are included, therefore, numerical solutions are significantly essential. In this paper, we will focus on the numerical simulation of some nonlinear dispersive water wave models with power-law nonlinearity.

Water wave dynamics are usually explained by the KdV equation (Korteweg-de Vries equation) [1–3], the RLW equation (Regularized Long-Wave equation) [4,5], and the Rosenau equation [6–8]. The KdV equation describes small-amplitude long-wave behavior on the surface of water in a channel. The RLW equation describes the undular bore behavior in water dynamics and simulates different situations of nonlinear dispersive waves for modeling a small-amplitude long wave in a channel. But the wave-wave and wave-wall interaction cannot be explained by the KdV and RLW equation. This can be fulfilled by the Rosenau equation because it is suitable for the dense discrete system dynamics. For further more understanding of nonlinear wave behavior, a viscous term $u_{xxx}$ and $u_{xxxx}$ needs to be included in the Rosenau equation, which brings the achievement of Rosenau-RLW, Rosenau-KDV, and Rosenau-KDV-RLW equation. In this paper, we focus on the generalized initial-boundary value problem of Rosenau-KDV-RLW equation with the homogeneous boundary conditions:

$$\begin{align*}
    \begin{cases}
    u_t + \delta u_{xxt} + \nu u_{xxxx} + \alpha u_x + \theta u_{xxx} + \varepsilon (u^p)_x = 0 & x \in (-\infty, +\infty), \ t \in (0, \infty) \\
    u(x, 0) = u_0(x) & x \in (-\infty, +\infty) \\
    u(\pm\infty, t) = u_x(\pm\infty, t) = u_{xx}(\pm\infty, t) = 0 & x \in (-\infty, +\infty)
    \end{cases}
\end{align*}$$

(1.1)

where $u(x, t)$ denote the wave profile, $x$ and $t$ are the spatial and temporal variables, respectively. $\alpha > 0$, $\varepsilon > 0$ are the parameters of linear and nonlinear advection term, $p \geq 2$ is the parameter of power-law nonlinearity. $\theta, \delta, \nu$ are the parameters of KdV, RLW, Rosenau terms, respectively. The equation (1.1) can be reduced to Rosenau-KdV equation with $\delta = 0$. 

2
and Rosenau-RLW equation with $\theta = 0$.

In the field, a considerable amount of literatures have been published to study the solitary wave behavior of these equations, both for theoretic and numerical analysis. A mass-preserving nonlinear method which combines a high-order compact scheme and a three-level average difference iterative algorithm, was analyzed and tested for the Rosenau-RLW equation in [9]. This equation was also investigated by a conservative three-level linear-implicit finite difference method in [10]. An attempt has been made to propose a conservative finite difference scheme for the Rosenau-RLW equation, which was unconditionally stable and the fourth-order in space and the second-order in time by Ahlem [11]. Another fourth-order accurate three-level conservative linear finite difference scheme was proposed by Hu [12] for the Rosenau-RLW equation.

The dynamics of dispersive shallow water wave that is governed by the Rosenau-KdV equation with power law nonlinearity was addressed in [13]. A conservative unconditionally stable finite difference scheme for 1D and 2D generalized Rosenau-KdV equation was proposed by Wang [14] with fourth and second-order accuracy in space and time, respectively. Wongsaijai [15] developed a three-level second-order accurate weighted average implicit finite difference scheme to solve the Rosenau-KdV equation and Rosenau-KdV-RLW equation. A three-level linear conservative implicit finite difference scheme was introduced by Wang [16] for solving the Rosenau-KdV-RLW equation. This equation was simulated efficiently by a multi-symplectic scheme and an energy-preserving scheme in [17] based on the multi-symplectic Hamiltonian formulation of the equation. The combination of a high efficient compact method for space and trapezoidal method for time integration was designed in [18], and numerical analysis is given for the effects of the parameters with Gaussian initial conditions on wave behavior of these type of equations.

The exponential time-differencing (ETD) method is an active and effective time solver with the advantages of solving linear part exactly and loosening the CFL restriction for stiff PDEs which contains linear and nonlinear terms, and applied generally for this type
of PDEs [19–22]. However, the original ETD Runge-Kutta (ETDRK) schemes [19] were proved to be unsuccessful because of disastrous cancellation error when the eigenvalue of the coefficient matrix in the linear part is equal to or close to zero, and also cannot be applied directly when the coefficient matrix in the linear term is non-diagonal. To overcome this difficulty, Kassam [20, 23, 24] came up with one effective solution, which was making use of Cauchy integral formula to deal with removable singularity mathematically exact and numerically accurate. The stability analysis of ETDRK scheme with Cauchy integral formula is provided in [22]. This point is very crucial in this paper because the linear coefficient matrix of the above model obtained by the compact finite difference method is always non-diagonal and has complex eigenvalues close to zero.

Since quite a wide range is taken as the computational domain of these type of long-wave models, always lead to the appearance of large-dimensional coefficient matrix in the computational process. Therefore, the dimension-reduction methods play an important role to reduce the computational cost. The Krylov subspace approximation [25,26] to the matrix exponential operator is an excellent choice in terms of both accuracy and efficiency to avoid unnecessary computational cost on the full exponential matrices since much smaller Krylov subspace is sufficient for large sparse coefficient matrix.

In section 2, we first briefly describe the fourth-order compact method in spatial discretization for (1.1). In section 3, the fourth-order ETD scheme, Cauchy integral formula, and Krylov subspace technique is given for the treatment in time direction. Stability analysis for the linear version of (1.1) is given in Section 4. Extensive numerical tests are shown in Section 5 to illustrate the accuracy and efficiency of the present method for nonlinear wave motions.

2 Spatial discretization

We consider Rosenau-KDV-RLW equation (1.1), and set computation domain as $[x_l, x_r]$. $\Delta x$ is the cell size of the equally-spaced grid in space. The uniform mesh is distributed as
follows:
\[ x \in [x_l, x_r], \quad \Delta x = \frac{x_r - x_l}{N - 1}, \quad x_j = x_l + (j - 1)\Delta x, \quad j = 1 : N. \]
The advection, convection, dispersion and Rosenau terms are denoted as:
\[ a \equiv u_x, \quad c \equiv u_{xx}, \quad d \equiv u_{xxx}, \quad r \equiv u_{xxxx}. \] (2.1)
The equation (1.1) can be rearranged as
\[ (u + \delta c + \nu r)_t = (-\alpha a - \theta d) - \varepsilon pu^{p-1}a. \] (2.2)
By discretizing the first and second-order spatial derivatives by means of three-point, fourth-order accurate compact method, we have
\[
\begin{align*}
\frac{1}{6}a_{j+1} + \frac{2}{3}a_j + \frac{1}{6}a_{j-1} &= \frac{1}{2\Delta x}(u_{j+1} - u_{j-1}), \\
\frac{1}{12}c_{j+1} + \frac{5}{6}c_j + \frac{1}{12}c_{j-1} &= \frac{1}{\Delta x^2}(u_{j+1} - 2u_j + u_{j-1}).
\end{align*}
\] (2.3)
Since the third and fourth-order spatial derivatives also can be written in terms of first and second-order derivatives as \( d = a_{xx} \) and \( r = c_{xx} \), we have
\[
\begin{align*}
\frac{1}{12}d_{j+1} + \frac{5}{6}d_j + \frac{1}{12}d_{j-1} &= \frac{1}{\Delta x^2}(a_{j+1} - 2a_j + a_{j-1}), \\
\frac{1}{12}r_{j+1} + \frac{5}{6}r_j + \frac{1}{12}r_{j-1} &= \frac{1}{\Delta x^2}(c_{j+1} - 2c_j + c_{j-1}).
\end{align*}
\] (2.4)
The homogeneous boundary condition for \( u(x, t) \) can be justified in finite computational domain as far as the wave is sufficiently far away from the boundaries and do not affect the solution in the interior domain. So We can easily derive the relationship between these derivatives from (2.3) and (2.4) with the homogeneous boundary conditions in (1.1):
\[ \tilde{A}A = BU, \quad \tilde{C}C = JU, \quad \tilde{C}D = JA, \quad \tilde{C}R = JC, \]
\[ \implies A = \tilde{A}^{-1}BU, \quad C = \tilde{C}^{-1}JU, \quad D = \tilde{C}^{-1}J\tilde{A}^{-1}BU, \quad R = \tilde{C}^{-1}J\tilde{C}^{-1}JU. \] (2.5)
where
\[
\tilde{A} = \begin{pmatrix}
\frac{2}{3} & \frac{1}{6} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}_{N \times N}, \quad B = \begin{pmatrix}
0 & \frac{1}{2\Delta x} & 0 \\
\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} \\
0 & \frac{1}{2\Delta x} & 0
\end{pmatrix}_{N \times N},
\]
\[
\tilde{C} = \begin{pmatrix}
\frac{5}{6} & \frac{1}{12} & 0 \\
\frac{1}{12} & \frac{5}{6} & \frac{1}{12} \\
0 & \frac{1}{12} & \frac{5}{6}
\end{pmatrix}_{N \times N}, \quad J = \begin{pmatrix}
\frac{-2}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 \\
\frac{1}{\Delta x^2} & 0 & \frac{1}{\Delta x^2} \\
0 & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2}
\end{pmatrix}_{N \times N},
\]
\[
W = \begin{pmatrix}
u_1^{p-1} & 0 \\
u_2^{p-1} & \vdots \\

\end{pmatrix}_{N \times N},
\]

and \(U = [u_1, u_2, \cdots, u_N]^T, A = [a_1, a_2, \cdots, a_N]^T, C = [c_1, c_2, \cdots, c_N]^T, D = [d_1, d_2, \cdots, d_N]^T, \)
\(R = [r_1, r_2, \cdots, r_N]^T.\) The whole system can be written in matrix form:
\[
\frac{dU}{dt} = LU + N_u(U).
\]
\[
(2.7)
\]

where \(L = (I + \delta \tilde{C}^{-1}J + \nu \tilde{C}^{-1}J \tilde{C}^{-1}J)^{-1}(-\alpha \tilde{A}^{-1}B - \theta \tilde{C}^{-1}J \tilde{A}^{-1}B), N_u(U) = (I + \delta \tilde{C}^{-1}J + \nu \tilde{C}^{-1}J \tilde{C}^{-1}J)^{-1}(-\varepsilon pW \tilde{A}^{-1}BU).\)

3 The forth order ETDRK-krylov method with Cauchy integral formula

The ordinary differential equation (2.7) has following formal solution as:
\[
U^{n+1} = \exp(\Delta tL)U^n + \int_0^{\Delta t} \exp((\Delta t - \tau)L)N_u\left(U(t_n + \tau)\right)d\tau,
\]
\[
(3.1)
\]
with
\[
\exp(\Delta tL) = \sum_{m=0}^{\infty} \frac{(\Delta tL)^m}{m!}, \quad N_u\left(U(t_n + \tau)\right) = \sum_{q=1}^{\infty} \frac{\tau^{q-1}}{(q-1)!} \frac{\partial^{q-1}N_u(U^n)}{\partial\tau^{q-1}}.
\]

6
where \( \Delta t \) is the temporal step size. This is the main path to propose exponential time difference method in which the stiff linear part is computed analytically whereas the nonlinear term is approximated numerically. ETDRK4 \([19]\) can be written as:

\[
U^{(1)} = \exp\left(\frac{L \Delta t}{2}\right)U^n + L^{-1}\left(\exp\left(\frac{L \Delta t}{2}\right) - I\right)N_u(U^n, t_n),
\]

\[
U^{(2)} = \exp\left(\frac{L \Delta t}{2}\right)U^n + L^{-1}\left(\exp\left(\frac{L \Delta t}{2}\right) - I\right)N_u(U^{(1)}, t_n + \frac{\Delta t}{2}),
\]

\[
U^{(3)} = \exp\left(\frac{L \Delta t}{2}\right)U^{(1)} + \left(\exp\left(\frac{L \Delta t}{2}\right) - I\right)\left(2N_u(U^{(2)}, t_n + \frac{\Delta t}{2}) - N_u(U^n, t_n)\right),
\]

\[
U^{n+1} = \exp(L \Delta t)U^n + \Delta t^{-1}\left(\left[-4I - \Delta tL + \exp(\Delta t L)(4I - 3\Delta tL + (\Delta tL)^2)\right]N_u(U^n, t_n) + \right.
\]

\[
\left[4I + 2\Delta tL + 2\exp(\Delta t L)(-2I + \Delta tL)\right]N_u(U^{(1)}, t_n + \frac{\Delta t}{2}) + N_u(U^{(2)}, t_n + \frac{\Delta t}{2}) + \right.
\]

\[
\left[-4I - 3\Delta tL - (\Delta tL)^2 + \exp(\Delta t L)(4I - \Delta t L)\right]N_u(U^{(3)}, t_n + \Delta t)\right).
\]

(3.2)

Here \( I \) is \( N \times N \) identity matrix. To reduce the numerical instability caused by cancellation error in high order ETD and Runge-Kutta ETD schemes, the modified ETD schemes are proposed in \([20]\) by using the Cauchy integral formula to evaluate the coefficients of the nonlinear part in (3.2) for ETDRK4, and also achieved the effort of generalizing the ETD schemes to non-diagonal problems. The Cauchy integral theory is introduced in detail in \([23, 24]\) while the stability analysis and truncation error analysis on modified ETD scheme with Cauchy integral formula are given in \([22]\).

The optimization technique for the coefficients of the nonlinear part is evaluating them over a contour \( C_o \) in the complex plane that encloses all the eigenvalues of \( L \) and is well separated from 0:

\[
f(L) = \frac{1}{2\pi i} \int_{C_o} f(t)(tI - L)^{-1} dt. \tag{3.3}
\]

The key to how this Cauchy integral works well in practice is to ensure that all eigenvalues of \( L \) are indeed enclosed by contour \( C_o \). For this nonlinear dispersive problem, the eigenvalues of \( L \) is always close to the imaginary axis. Here we take the simplest approach in which the contour \( C_o \) is a circle of radius \( r_o = \max(\{|\text{eig}(L)|\}) \) with angles \( \theta = \left\{\frac{\pi}{32}, \frac{3\pi}{32}, \cdots, \frac{63\pi}{32}\right\} \) sampled at equally spaced points \( z = r_o \theta \).
We consider the coefficients in the update formula (3.2) of ETDRK4 and let \( Q = \Delta t L \):

\[
M_1 = L^{-1} \left( \exp \left( \frac{L \Delta t}{2} \right) - I \right) = \Delta t Q^{-1} \left( \exp \left( \frac{Q}{2} \right) - I \right),
\]

\[
M_2 = \Delta t^{-2} L^{-3} \left[ -4I - \Delta t L + \exp(\Delta t L) (4I - 3\Delta t L + (\Delta t L)^2) \right]
= \Delta t Q^{-3} \left( -4I - Q + \exp(Q)(4I - 3Q + Q^2) \right),
\]

\[
M_3 = \Delta t^{-2} L^{-3} \left[ 4I + 2\Delta t L + 2 \exp(\Delta t L)(-2I + \Delta t L) \right]
= \Delta t Q^{-3} (4I + 2Q + 2 \exp(Q)(-2I + Q)),
\]

\[
M_4 = \Delta t^{-2} L^{-3} \left[ -4I - 3\Delta t L - (\Delta t L)^2 + \exp(\Delta t L)(4I - \Delta t L) \right]
= \Delta t Q^{-3} \left( -4I - 3Q - Q^2 + \exp(Q)(4I - Q) \right).
\]

The modification of these coefficients in above complex circle is given in [20, 23]:

\[
M_1(Q) = \frac{1}{2\pi i} \int_{C_0} M_1(z)(zI - Q)^{-1} dz = \frac{1}{2\pi} \int_0^{2\pi} \Delta t \left( \exp \left( \frac{z}{2} \right) - 1 \right)(zI - Q)^{-1} d\theta,
\]

\[
M_2(Q) = \frac{1}{2\pi i} \int_{C_0} M_2(z)(zI - Q)^{-1} dz = \frac{1}{2\pi} \int_0^{2\pi} \Delta t \left( -4 - z + \exp(z)(4 - 3z + z^2) \right)(zI - Q)^{-1} d\theta,
\]

\[
M_3(Q) = \frac{1}{2\pi i} \int_{C_0} M_3(z)(zI - Q)^{-1} dz = \frac{1}{2\pi} \int_0^{2\pi} \Delta t \left( 4 + 2z + 2 \exp(z)(-2 + z) \right)(zI - Q)^{-1} d\theta,
\]

\[
M_4(Q) = \frac{1}{2\pi i} \int_{C_0} M_4(z)(zI - Q)^{-1} dz = \frac{1}{2\pi} \int_0^{2\pi} \Delta t \left( -4 - 3z - z^2 + \exp(z)(4 - z) \right)(zI - Q)^{-1} d\theta.
\]

(3.5)

we have

\[
q_1(z) = \Delta t \left( \exp \left( \frac{z}{2} \right) - 1 \right)(zI - Q)^{-1},
\]

\[
q_2(z) = \Delta t \left( -4 - z + \exp(z)(4 - 3z + z^2) \right)(zI - Q)^{-1},
\]

\[
q_3(z) = \Delta t \left( 4 + 2z + 2 \exp(z)(-2 + z) \right)(zI - Q)^{-1},
\]

\[
q_4(z) = \Delta t \left( -4 - 3z - z^2 + \exp(z)(4 - z) \right)(zI - Q)^{-1}.
\]

(3.6)

finally, we obtain the modified version of these coefficients in complex plane by the means of trapezoidal rule

\[
M_1(Q) = \frac{1}{32} \sum_{s=1}^{32} q_1(z_s), \quad M_2(Q) = \frac{1}{32} \sum_{s=1}^{32} q_2(z_s), \quad M_3(Q) = \frac{1}{32} \sum_{s=1}^{32} q_3(z_s), \quad M_4(Q) = \frac{1}{32} \sum_{s=1}^{32} q_4(z_s).
\]

(3.7)

Exponential computation for matrix L in the linear part always causes enormous computational cost and uses excessive amounts of compute time. For the propose of improving this
situation, we choose to use Krylov subspace technique [25, 26] to approximate \( \exp(\Delta t L) U \), so we do not need the full exponential matrices \( \exp(\Delta t L) \) but only the products of it and orthonormal basis of the Krylov subspace. First, we define \( k \)-dimensional Krylov subspace and orthonormal basis matrix for large sparse matrix \( L \) as:

\[
K_{L,U} = \text{span}\{ U, LU, L^2U, \cdots, L^{k-1}U \}, \quad V_k = (v_1, v_2, \cdots, v_k).
\]  

(3.8)

The orthonormal basis matrix \( V_k \in \mathbb{R}^{N \times k} \) satisfies well-known Arnoldi decomposition and satisfies

\[
LV_k = V_{k+1}^T \tilde{H}_k.
\]  

(3.9)

where \( V_{k+1} = (v_1, v_2, \cdots, v_k, v_{k+1}) \in \mathbb{R}^{N \times (k+1)} \) and \( \tilde{H}_k \) is the \((k + 1) \times k\) upper-Hessenberg matrix has the form:

\[
\tilde{H}_k = \begin{bmatrix}
H_k & h_{k+1,k}e_k^T \\
h_{k+1,k}e_k^T & 0
\end{bmatrix}_{k+1 \times k}.
\]  

(3.10)

where \( H_k \) is the matrix composed of the first \( k \) rows of \( \tilde{H}_k \), and \( e_k = (0, \cdots, 0, 1)^T \in \mathbb{R}^k \) is the \( k \)-th canonical basis vector in \( \mathbb{R}^k \), then (3.9) becomes

\[
LV_k = V_k^T H_k + h_{k+1,k}V_{k+1}^T e_k.
\]  

(3.11)

Because \( V_k^T V_k = I \), therefore

\[
H_k = V_k^T L V_k.
\]  

(3.12)

Since \( H_k \) is the projection of the linear transformation of \( L \) onto the subspace \( K_k \) with the basis \( V_k \), and \( V_k V_k^T \neq I \), then (3.11) leads to the following approximation:

\[
L \approx V_k V_k^T L V_k V_k^T = V_k^T H_k V_k^T.
\]  

(3.13)

and \( \exp(\Delta t L) U \) can be approximated as follows:

\[
\exp(\Delta t L) U \approx \exp(\Delta t V_k^T H_k V_k^T) U = V_k \exp(\Delta t H_k) V_k^T U.
\]  

(3.14)

The first column vector of \( V_k \) is taken as \( v_1 = U/\|U\| e_1 \) and \( V_k^T U = \|U\|e_1 \), thus (3.14) becomes:

\[
\exp(\Delta t L) U \approx \|U\|_2 V_k \exp(\Delta t H_k) e_1.
\]  

(3.15)
finally, the final fourth-order ETD Runge-Kutta scheme takes the form as:

\[
U^{(1)} = ||U^n||_2 V_k \exp \left( \frac{\Delta t H_k}{2} \right) e_1 + M_1(Q) N_u(U^n, t_n),
\]

\[
U^{(2)} = ||U^n||_2 V_k \exp \left( \frac{\Delta t H_k}{2} \right) e_1 + M_1(Q) N_u(U^{(1)}, t_n + \frac{\Delta t}{2}),
\]

\[
U^{(3)} = ||U^{(1)}||_2 V_k \exp \left( \frac{\Delta t H_k}{2} \right) e_1 + M_1(Q) \left( 2 N_u(U^{(2)}, t_n + \frac{\Delta t}{2}) - N_u(U^n, t_n) \right),
\]

\[
U^{n+1} = ||U^n||_2 V_k \exp(\Delta t H_k) e_1 + \left[ M_2(Q) N_u(U^n, t_n) + M_3(Q) \left( N_u(U^{(1)}, t_n + \frac{\Delta t}{2}) + N_u(U^{(2)}, t_n + \frac{\Delta t}{2}) \right) + M_4(Q) N_u(U^{(3)}, t_n + \Delta t) \right].
\]

The dimension of the Krylov subspace \( k \) is much smaller than the dimension of the large sparse matrix \( L \). For the computational process in this paper, we take one efficient \( k \) depend on the dimension of \( L \) to make sure the Arnoldi algorithm [25,26] does not cause extra error in numerical computation.

### 4 Linear stability of the fourth order ETD Runge-Kutta method with Cauchy integral

Here we give stability property of (3.16) for the linear scalar version of (1.1) with \( p = 1 \) while ignoring the Arnoldi Algorithm. Use of \( u_j = \bar{u} \exp(ik_j \Delta x) \), \( a_j = \bar{a} \exp(ik_j \Delta x) \), \( c_j = \bar{c} \exp(ik_j \Delta x) \) at \((x_j, t)\), the Fourier relationship between varies spacial derivatives in (2.3) and (2.4) are:

\[
\bar{a} = \bar{u} \frac{i \sinh \Delta x}{\Delta x (\frac{1}{3} \cosh \Delta x + \frac{2}{3})}, \quad \bar{c} = \bar{u} \frac{2 \cosh \Delta x - 2}{\Delta x^2 (\frac{1}{6} \cosh \Delta x + \frac{2}{3})},
\]

\[
\bar{d} = \bar{a} \frac{2 \cosh \Delta x - 2}{\Delta x^2 (\frac{1}{6} \cosh \Delta x + \frac{2}{3})}, \quad \bar{r} = \bar{c} \frac{2 \cosh \Delta x - 2}{\Delta x^2 (\frac{1}{6} \cosh \Delta x + \frac{2}{3})}.
\]

The scalar linear version of (2.7) yields

\[
\bar{u}_t = l \bar{u} + \lambda \bar{u}.
\]

where

\[
l = -\alpha \frac{i \sinh \Delta x}{\Delta x (\frac{1}{3} \cosh \Delta x + \frac{2}{3})} - \theta \frac{i \sinh \Delta x}{\Delta x (\frac{1}{3} \cosh \Delta x + \frac{2}{3})} \frac{2 \cosh \Delta x - 2}{\Delta x^2 (\frac{1}{6} \cosh \Delta x + \frac{2}{3})},
\]

\[
\lambda = -\frac{\varepsilon}{\Delta x (\frac{1}{3} \cosh \Delta x + \frac{2}{3})} - \frac{1}{\Delta x^2 (\frac{1}{6} \cosh \Delta x + \frac{2}{3})} + \nu \left( \frac{2 \cosh \Delta x - 2}{\Delta x^2 (\frac{1}{6} \cosh \Delta x + \frac{2}{3})} \right)^2.
\]
Recall the approximation of the two-point quadrature amount in scalar case:

\[
\frac{\exp \left( \frac{l \Delta t}{2} \right) - 1}{l} \sim \frac{1}{2} \left( \exp \left( \frac{(l+r_o) \Delta t}{2} \right) - 1 + \exp \left( \frac{(l-r_o) \Delta t}{2} \right) - 1 \right),
\]

where \( r_o \) and \(-r_o\) are so chosen as the quadrature points for the approximation of the contour integral in contour with radius \( r_o \) and centered at 0 in this paper. Simplify the right hand side of (4.3) and let:

\[
\frac{\exp \left( \frac{l \Delta t}{2} \right) - 1}{l} \sim \frac{l \exp \left( \frac{l \Delta t}{2} \right) \cos \left( \frac{r_o \Delta t}{2} \right) + r_o \exp \left( \frac{l \Delta t}{2} \right) \sin \left( \frac{r_o \Delta t}{2} \right) - l}{l^2 + r_o^2} = \phi.
\]

From (3.2), we can write \( \bar{u}^{(1)}, \bar{u}^{(2)}, \bar{u}^{(3)} \) and the corresponding amplification factors as:

\[
\begin{align*}
\bar{u}^{(1)} &= \left( \exp \left( \frac{l \Delta t}{2} \right) + \phi \lambda \right) \bar{u}, & g^{(1)} &= \exp \left( \frac{l \Delta t}{2} \right) + \phi \lambda, \\
\bar{u}^{(2)} &= \left( \exp \left( \frac{l \Delta t}{2} \right) + \phi \lambda g^{(1)} \right) \bar{u}, & g^{(2)} &= \exp \left( \frac{l \Delta t}{2} \right) + \phi \lambda g^{(1)}, \\
\bar{u}^{(3)} &= \left( \exp \left( \frac{l \Delta t}{2} \right) g^{(1)} + \phi \lambda (2g^{(2)} - 1) \right) \bar{u}, & g^{(3)} &= \exp \left( \frac{l \Delta t}{2} \right) g^{(1)} + \phi \lambda (2g^{(2)} - 1).
\end{align*}
\]

Using these, we finally can derive the corresponding amplification factor

\[
G(l, \lambda) = \exp(l \Delta t) + \varphi_1 \lambda + \varphi_2 \lambda(g^{(1)} + g^{(2)}) + \varphi_3 \lambda g^{(3)}.
\]

here \( \varphi_1 = \frac{-4-4l \Delta t+(\phi+1)^2(4-3l \Delta t+2l^2)}{2l^2+4l^2}, \varphi_2 = \frac{4+2l \Delta t+2(\phi+1)^2(-2l \Delta t)}{2l^2+4l^2}, \varphi_3 = \frac{-4-3l \Delta t-3l^2+2(\phi+1)^2(l-3 \Delta t)}{2l^2+4l^2}. \)

In [22], significant analysis is given in the case of \( l \) is negative real while \( \lambda \) is complex, and \( l, \lambda \) are both real. Notice that in our case \( l \) and \( \lambda \) are both complex, which still remains as a open problem. Unfortunately, we do not know any expression for \( |G(l, \lambda)| \leq 1 \).

To demonstrate the stability prosperity of the scheme, we compute the amplification factor \( |G(l, \lambda)| \) for varies \( CFL \) and \( \Delta x \) for the linear version of (1.1) with parameter \( \delta = -1, \nu = 1, \theta = 1, \alpha = 0, \varepsilon = 1, p = 1 \). Notice that in computing \( \phi, g^{(1)}, g^{(2)}, g^{(3)}, \varphi_1, \varphi_2, \varphi_3 \) suffers from numerical instability for \( l \) and \( \Delta t \) close to zero. Because of that, we will use their five-term Taylor expansions in computational process. In Figure 4.1(left), we take \( CFL = [0 : 0.1 : 10], \Delta x = [10^{-10} : 10^{-1} : 1] \) and \( \Delta t = CFL \cdot \Delta x \). It can be observed that \( |G(l, \lambda)| \) gets bigger than 1 with small increase along with the increase in \( CFL \) and \( \Delta x \). Figure 4.1(right) is the partial graph for \( CFL = [0 : 0.1 : 1], \Delta x = [10^{-10} : 10^{-1} : 1] \).
In this paper, we take $CFL = 1, \Delta x = \Delta t \leq 0.4$ for all numerical simulation to ensure the numerical stability of the scheme for equation (1.1).

5 Numerical results

In this section, The algorithm (3.16) is applied through the Rosenau-RLW equation, Rosenau-KdV equation, and Rosenau-KdV-RLW equation to confirm the performance of the present method. The advantages of the ETD method allow us to take a large time step, so that $\Delta t = CFL \cdot \Delta x$ with $CFL = 1$ is sufficiently enough for rates of convergence in terms of $L_2$ and $L_\infty$ for each test cases. We observe during the numerical experiment that the computational error remains stable for Krylov subspace dimension $k$ bigger than one positive integer that much smaller than the dimension of the large sparse matrix $L$. In the following numerical tests, we take different $k$ for each examples depending on the dimension of $L$, and list it along with the numerical results.

Example 1. Consider Rosenau-RLW equation (1.1) with parameters $\delta = -1, \nu = 1, \alpha = 0, \varepsilon = 1, p = 1$:

$$u_t - u_{xxt} + u_{x,xxt} + u_x + \frac{1}{2}(u^2)_x = 0, \quad x \in [-50, 150], t \in [0, T].$$
with the initial condition \( u(x, 0) = \frac{15}{19} \text{sech}^4\left(\frac{\sqrt{13}}{26}x\right) \) and the analytical solitary wave solution 
\( u(x, t) = \frac{15}{19} \text{sech}^4\left(\frac{\sqrt{13}}{26}(x - \frac{169}{133}t)\right) \).

Numerical solutions on various mesh sizes are calculated and compared with the exact solution of the Rosenau-RLW equation. We model a solitary wave approximately and present the errors and rates of convergency in terms of \( L_2 \) and \( L_\infty \) of the simulation at \( T = 24 \) for 
\( \Delta t = CFL \cdot \Delta x \) with \( CFL = 1 \) and corresponding Krylov subspace dimension \( k \) over interval 
\( x \in [-50, 150] \). Numerical results are compared with previously published studies assessed in the introduction on Rosenau-RLW equation in Table 5.1. We can see that experimental results agree with the theoretical fourth-order convergency rate in the case \( \Delta x = \Delta t \).

The Figure 5.1(left) shows that the solitary wave curve matches excellently with exact solution when \( \Delta x = \Delta t = 0.1 \) with \( k = 30 \) at \( T = 24 \) over interval \( x \in [-50, 150] \). From the Figure 5.1(right), it can be seen that error mostly generates at two sides of the solitary wave peak. A right-propagating waves with almost identical amplitude are generated in the Figure 5.2 at different times \( T = 0, 10, 20 \). Solitary Wave moves far away from its generation location through time with unchanged shape.

Table 5.1: The errors and rates of convergence of numerical solutions at \( T = 24 \) with \( CFL = 1, \Delta x = \Delta t \) in \( x \in [-50, 150] \) for Example 1.

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<th>( \text{Ref [15]}(\frac{-1}{3}) )</th>
<th>( \text{Ref [9]} )</th>
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| \( \Delta x \) | \( L_\infty \) | \( L_\infty \) | \( L_\infty \) | \( L_\infty \) |
|----------------|----------------|----------------|----------------|
| \( L_\infty \) | \( L_\infty \) | \( L_\infty \) | \( L_\infty \) |
| 0.4 | 10 | 2.4843e-02 | 1.8240e-03 | 1.8720e-05 | 1.1827e-05 |
| 0.2 | 20 | 6.3640e-03 | 1.9673 | 4.6918e-04 | 1.9589 | 1.1652e-06 | 4.0059 | 1.1361e-07 | 4.0083 |
| 0.1 | 30 | 1.1298e-04 | 1.9905 | 1.1827e-04 | 1.9881 | 7.2777e-08 | 4.0025 | 4.5947e-08 | 3.9996 |
| 0.05 | 40 | 2.8310e-05 | 1.9967 | 2.9691e-05 | 1.9940 | 5.2026e-09 | 3.8062 | 2.9034e-09 | 3.9841 |
Figure 5.1: Numerical solution of Rosenau-RLW equation with $CFL = 1, \Delta x = \Delta t = 0.1, k = 30$ at $T = 24$ in $x \in [-50, 150]$ (left) and error (right) for Example 1.

Figure 5.2: Wave graph of Rosenau-RLW equation with $CFL = 1, \Delta x = \Delta t = 0.1, k = 30$ at $T = 0, 10, 20$ in $x \in [-50, 150]$ for Example 1.

Figure 5.3: Numerical solution of Rosenau-KdV equation with $CFL = 1, \Delta x = \Delta t = 0.1, k = 30$ at $T = 20$ in $x \in [-70, 100]$ for Example 2.
Example 2. Consider Rosenau-KdV equation (1.1) with parameters $\delta = 0, \nu = 1, \alpha = 1, \theta = 1, \varepsilon = \frac{1}{2}, p = 2$:

$$u_t + u_{xxxx}t + u_{xx} + u_x + \left(\frac{1}{2}u^2\right)_x = 0, \quad x \in [-70, 100], t \in [0, T].$$

Rosenau-KdV equation has analytical solitary wave solution $u(x, t) = M \text{sech}^{\frac{1}{4}}[W(x - Vt)]$ as in [13] with wave width $W = \frac{p-1}{p+1} \left[\frac{-\alpha \nu (p^2+2p+5) + \sqrt{\alpha^2 \nu^2 (p^2+2p+5)^2 + 16 \theta^2 \nu (p+1)^2}}{32 \nu \theta} \right]^{\frac{1}{2}}$, wave amplitude $M = \left[\frac{-\alpha \nu (p^2+2p+5) + \sqrt{\alpha^2 \nu^2 (p^2+2p+5)^2 + 16 \theta^2 \nu (p+1)^2}}{16 \nu (p+1)(p^2+2p+5)} \right]^{\frac{1}{2}}$, and wave velocity $V = \theta \frac{p-1}{4 \nu W (p^2+2p+5)}$. Here we take the initial condition to be $u(x, 0) = M \text{sech}^{\frac{1}{4}}(Wx)$.

We simulate this solitary wave problem under various mesh. The calculated errors, rates of convergency and corresponding Krylov subspace dimension $k$ in terms of $L_2$ and $L_\infty$ at $T = 20$ for $\Delta t = CFL \cdot \Delta x$ with $CFL = 1$ over interval $x \in [-70, 100]$ are listed in Table 5.2. It can be observed that the present scheme has the smallest error with fourth-order convergency than the other three different numerical schemes.

The solution profile and distribution of absolute error are shown in the right and left of Figure 5.3 for $\Delta x = \Delta t = 0.1$ with $k = 30$ at time $T = 20$. It seems that the maximum error occurs near the peak amplitude of the solitary wave. The initial solitary wave at $T = 0$ and numerical solutions at $T = 10, 20$ with $\Delta x = \Delta t = 0.1$ are given in Figure 5.4.

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Example 3. Consider Rosenau-KdV-RLW equation (1.1) with parameters \( \delta = -1, \nu = 1, \alpha = 1, \theta = 1, \varepsilon = \frac{1}{2}, p = 2 \):

\[
    u_t - u_{xxx} + u_{xxxx} + u_x + u_{xxx} + \frac{1}{2}(u^2)_x = 0, \quad x \in [-40, 60], t \in [0, T].
\]

Rosenau-KdV-RLW equation has solitary wave solution \( u(x, t) = M \text{sech}^{4/p-1}(Wx) \) with

\[
    S = \sqrt{\alpha^2 \nu^2(p^2 + 2p + 5)^2 + 16(p + 1)^2 \theta \nu(\theta - \alpha \delta)}, \quad \text{wave width} \quad W = \frac{p-1}{p+1} \sqrt{\frac{S-(p^2+2p+5)\alpha \nu}{32 \theta \nu}},
\]

wave speed \( V = \frac{\theta(p-1)^2}{(p-1)^2 \delta + 4 \nu W^2(p^2 + 2p + 5)} \), and amplitude \( M = \left[ \frac{8(p+1)(p+3)(3p+1)\theta \nu W^4}{e(p-1)^2((p-1)^2 \delta + 4(p^2 + 2p + 5)\nu W^2)} \right]^{\frac{1}{p-1}}. \)

Here we choose the initial condition to be \( u(x, 0) = M \text{sech}^{4/p-1}(Wx) \).

We report the calculated errors, rates of convergency and corresponding Krylov subspace dimension \( k \) in terms of \( L_2 \) and \( L_\infty \) at \( T = 10 \) for \( \Delta t = CFL \cdot \Delta x \) with \( CFL = 1 \) over interval \( x \in [-40, 60] \) in Table 5.3. We make comparison on the computation errors and convergency rates between with three different numerical schemes to illustrate the superiority of present scheme (3.16).

Figure.5.5 (Left) presents numerical solitary wave profile for \( \Delta x = \Delta t = 0.1 \) with \( k = 30 \) at time \( T = 10 \), which is close to analytical values. Figure.5.5 (right) presents the corresponding distribution of absolute errors at time \( T = 10 \). Furthermore, Figure.5.6 indicates that the right-propagating solitary wave profile travels with unchanged shape and amplitude of the same height through time \( T = 0, 10, 20 \) under \( \Delta x = \Delta t = 0.1 \).

6 Concluding Remark

The fourth-order compact finite difference method for spatial discretization and fourth-order ETD Runge-Kutta method with Cauchy integral application for time discretization is applied for some nonlinear dispersive wave equations. The Cauchy integral formula plays a vital role in stabilizing the numerical method for this type of nonlinear dispersion equations in the case of non-diagonal large sparse coefficient matrix with extremely small complex eigenvalues. Stability analysis is given by computation in the case of complex linear and nonlinear coefficient. Arnoldi algorithm is considered to reduce the computational cost on
Figure 5.4: Wave graph of Rosenau-KdV equation with $CFL = 1, \Delta x = \Delta t = 0.1, k = 30$ at $T = 0, 10, 20$ in $x \in [-70, 100]$ for Example 2.

Figure 5.5: Numerical solution of Rosenau-KDV-RLW equation with $CFL = 1, \Delta t = \Delta x = 0.1, k = 20$ at $T = 10$ in $x \in [-40, 60]$ (left) and error (right) for Example 3.

Figure 5.6: Wave graph of Rosenau-KDV-RLW equation with $CFL = 1, \Delta t = \Delta x = 0.1, k = 20$ at $T = 0, 10, 20$ in $x \in [-40, 60]$ for Example 3.
Table 5.3: The errors and rates of convergence of numerical solutions at $T = 10$ with $CFL = 1, \Delta t = \Delta x$ in $x \in [-40, 60]$ for Example 3.

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the exponential part. Fourth-order accuracy is achieved in both space and time theoretically and also varied numerically. We mainly focus on simulation of the solitary wave of Rosenau-RLW equation, Rosenau-KdV equation, and Rosenau-KdV-RLW equation. The present scheme approximates every test case commendably with quit large step size in time and space direction. Numerical simulations indicated that the present method is very efficient with loosely-restricted CFL condition.

References


