A new third order finite volume weighted essentially non-oscillatory scheme on tetrahedral meshes

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\textbf{A B S T R A C T}

In this paper a third order finite volume weighted essentially non-oscillatory scheme is designed for solving hyperbolic conservation laws on tetrahedral meshes. Comparing with other finite volume WENO schemes designed on tetrahedral meshes, the crucial advantages of such new WENO scheme are its simplicity and compactness with the application of only six unequal size spatial stencils for reconstructing unequal degree polynomials in the WENO type spatial procedures, and easy choice of the positive linear weights without considering the topology of the meshes. The original innovation of such scheme is to use a quadratic polynomial defined on a big central spatial stencil for obtaining third order numerical approximation at any points inside the target tetrahedral cell in smooth region and switch to at least one of five linear polynomials defined on small biased/central spatial stencils for sustaining sharp shock transitions and keeping essentially non-oscillatory property simultaneously. By performing such new procedures in spatial reconstructions and adopting a third order TVD Runge–Kutta time discretization method for solving the ordinary differential equation (ODE), the new scheme’s memory occupancy is decreased and the computing efficiency is increased. So it is suitable for large scale engineering requirements on tetrahedral meshes. Some numerical results are provided to illustrate the good performance of such scheme.

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1. Introduction

In this paper we design a new third order finite volume weighted essentially non-oscillatory (WENO) to solve three dimensional hyperbolic conservation laws

\[
\begin{align*}
\frac{\partial u}{\partial t} + f(u) \cdot \nabla x + g(u) \cdot \nabla y + r(u) \cdot \nabla z &= 0, \\
\mathbf{u}(x, y, z, 0) &= \mathbf{u}_0(x, y, z),
\end{align*}
\]  

(1.1)
on tetrahedral meshes. As it is well known, the essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes are popular high order numerical methods for solving hyperbolic conservation laws (1.1) with strong shocks, contact disconti-

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nities and various smooth structures. Harten et al. proposed ENO schemes \([18,32,33]\). In 1994, Liu, Osher and Chan \([24]\) firstly presented a first third order finite volume WENO scheme. In 1996, Jiang and Shu \([20]\) designed fifth order finite difference WENO scheme in one and two dimensions with a general framework for the design of the smoothness indicators and associated nonlinear weights. Then some classical finite volume WENO schemes were constructed \([15,19,22,26,30]\) on structured and unstructured meshes obeying the similar reconstruction principles. Both ENO and WENO schemes used the idea of adaptive stencils to automatically achieve high order accuracy in smooth region and keep non-oscillatory property near discontinuities. In comparisons, the main spirit of the WENO schemes which were superior to the ENO schemes was that they reconstructed a convex combination relationship of all equal degree polynomials which were defined on the information of different equal size spatial stencils and used it as the building block for the final WENO reconstructions. By the applications of the optimal linear weights and associated smoothness indicators, the nonlinear weights were obtained \([20,31]\).

Although ENO and WENO schemes \([6,7,16]\) had obvious advantages, it was more difficult to design WENO schemes on unstructured meshes, for examples, triangular meshes (in 2D) or tetrahedral meshes (in 3D). Classical two dimensional third order and fourth order finite volume WENO schemes were proposed by Hu and Shu \([19]\) on triangular meshes. In which, they gave a new way of measuring two dimensional smoothness of numerical solutions which was different to the expressions specified in \([1]\) and \([15]\). But the skills of maintaining positive linear weights for the high order finite volume WENO schemes were too complex to be fulfilled and such circumstance resulted in the hard of engineering applications especially in three dimensions \([38]\). There are two types of different WENO schemes designed in the literature on unstructured meshes. The first type includes some so called robust WENO schemes whose order of accuracy was not higher than each of the reconstruction defined on associated spatial stencil. For this type of WENO schemes, the linear weights were artificially set without precisely calculating and obeying some reconstruction principles, and associated nonlinear weights were designed purely to avoid spurious oscillations and keep stability. By doing so, such WENO schemes did not offer some contributions to increase the order of accuracy and might decay the convergence rate of some steady state problems. Since the linear weights could be set as any arbitrary positive constants, such robust but low order accurate WENO schemes \([11,12,15,35]\) were easier to be constructed and suitable for engineering applications. Another type includes some classical WENO schemes \([19,30,38]\) et al. whose order of accuracy was much higher than that of the reconstruction on each spatial stencil. For example, the third order and fourth order WENO schemes \([19]\) were based on nine linear polynomials defined on three-cell spatial stencils, and on six quadratic polynomials defined on six-cell spatial stencils. This type of WENO schemes were more difficult to be constructed for the sake of obtaining their optimal linear weights at different quadrature points on the boundaries of the target cell. But they had a more compact stencil than the former type of WENO schemes to obtain the same order accuracy.

If the computing meshes distort greatly or change via time approaching (such as for the space–time adaptive meshes \([10,13,37]\) or Arbitrary-Lagrange–Euler (ALE) problems \([4]\) et al.), we should deal with the occurrence of the negative linear weights \([30]\) and cost a lot of time in computing the optimal linear weights by solving a linear system at different quadrature points on the boundaries of the target cell or cost many random-access memories to store them. Liu and Zhang \([25]\) found that it was hard to design a robust WENO scheme when facing distorted local mesh geometries or degenerate cases when the meshes’ quality varied for a complex domain geometry. If the linear weights were negative greatly or non-existent in such circumstances, it would tremendously destroy the good performance of the finite volume WENO schemes. When there are no linear weights at some specific quadrature points on the boundaries of the target cell, the WENO reconstruction procedures will fail to increase the order of accuracy to the optimal one. Such problem of non-existent linear weights for third order WENO reconstruction at the center of the cell was addressed in the central/compact WENO (CWENO) schemes \([2,5,9,21–23,29]\). So as for the purpose of overcoming the drawback of WENO spatial reconstruction that the linear weights may not exist, we developed a new fifth order finite difference WENO scheme \([40]\) with one five-point stencil and two two-point stencils. Following the idea of such classical CWENO schemes \([22,23]\) and the new finite difference/volume WENO scheme \([40–42]\), we extend the new WENO schemes from the structured meshes to three dimensional unstructured meshes.

After the analysis of ENO and WENO schemes in detail, a class of new, simple and robust finite volume WENO schemes was presented on structure and unstructured meshes \([41,42]\). In this paper, we extend such method to three dimensional cases on tetrahedral meshes. The basic flowcharts are briefly narrated as follows. We select a big central spatial stencil which contains no less than ten tetrahedral cells including the target cell, and then reconstruct a quadratic polynomial based on the information of the conservative variables defined on each tetrahedral cell. Hereafter, we select five five-cell small biased/coronal spatial stencils including the target cell and reconstruct five linear polynomials which equal the cell average on the target cell and match the cell averages on the other cells in a least square sense \([19]\). The quadratic polynomial needs to be modified at different quadrature points on the boundaries of the target cell, so as to keep third order approximation in smooth region \([19,31,38]\). After performing these modifications, any positive linear weights could be randomly set in case their summation is one. After the computation of the smoothness indicators, the application of a new formulation of the nonlinear weights and the third order TVD Runge–Kutta time discretization method \([32]\), the new third order finite volume WENO scheme is obtained both in space and time. We should point out that such new scheme also obeys the principles proposed by Sonar \([34]\), Harten and Chakravarthy \([17]\), and Vankeirsbilck \([36]\) on tetrahedral meshes.
2. Description of the third order WENO scheme

We consider three dimensional conservation laws (1.1) on tetrahedral meshes and integrate (1.1) over the target cell \( \Delta_0 \) to get the semi-discrete finite volume formulation

\[
\frac{du(t)}{dt} = -\frac{1}{|\Delta_0|} \int_{\partial \Delta_0} F \cdot n_{\Delta} ds = L(u),
\]

(2.1)

where \( u(t) = \frac{1}{|\Delta_0|} \int_{\Delta_0} u(x, y, z, t) dxdydz, F = (f, g, r), \partial \Delta_0 \) is the boundary of the target cell \( \Delta_0, |\Delta_0| \) is the volume of the target cell \( \Delta_0 \) and \( n_{\Delta} = (n_x, n_y, n_z)^T \) denotes the outward unit normal to the boundary of the target cell \( \Delta_0 \). The surface integrals in (2.1) are discretized by a six-point quadrature integration formula on every triangular element (for example, for a triangular element with three vertices \( (x_1, y_1, z_1), (x_2, y_2, z_2) \) and \( (x_3, y_3, z_3) \), the six-point quadrature points are \((x_{G1}, y_{G1}, z_{G1}) = (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3, \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3), (x_{G2}, y_{G2}, z_{G2}) = (\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3, \beta_1 y_1 + \beta_2 y_2 + \beta_3 y_3, \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3), (x_{G3}, y_{G3}, z_{G3}) = (\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3, \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3, \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3),\)

\[
\int_{\partial \Delta_0} F \cdot n_{\Delta} ds \approx \sum_{\ell = 1}^{6} \frac{|\partial \Delta_{0\ell}|}{|\Delta_0|} \sum_{\ell = 1}^{4} \sigma_{\ell} F(u(x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}, t)) \cdot n_{\ell \ell}.
\]

(2.2)

And \( F(u(x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}, t)) \cdot n_{\ell \ell}, \ell = 1, \ldots, 6, \ell \ell = 1, 2, 3, 4 \) are reformulated by numerical fluxes such as the Lax–Friedrichs flux

\[
1 \left( (F(u^+(x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}, t)) + F(u^-(x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}, t))) \cdot n_{\ell \ell} - \right.
\]

\[
\alpha(u^+(x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}, t) - u^-(x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}, t)) \right), \ell = 1, \ldots, 6, \ell \ell = 1, 2, 3, 4.
\]

(2.3)

Here \( \alpha \) is taken as an upper bound for the eigenvalues of the Jacobian in the \( n_{\ell \ell} \) direction, and \( u^+ \) and \( u^- \) are the conservative values of \( u \) inside and outside the boundaries of the target tetrahedral cell (inside of the neighboring tetrahedral cell) at different quadrature points and \( |\partial \Delta_{0\ell}|, \ell = 1, 2, 3, 4 \) are the area of the triangular elements. We then emphasize the procedures of a new third order finite volume WENO scheme on tetrahedral meshes as follows and omit variable \( t \) if not cause confusion, unless specified otherwise.

The reconstruction of function \( u(x, y, z, t) \) at different quadrature points \( (x_{G_{\ell}}, y_{G_{\ell}}, z_{G_{\ell}}) \), \( \ell = 1, \ldots, 6, \ell \ell = 1, 2, 3, 4 \) on the boundaries of target cell \( \Delta_0 \) is narrated as follows. For simplicity, we use the spatial stencils in Fig. 2.1.

Step 1. Select the big central spatial stencil as \( T_1 = \{ \Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{31}, \Delta_{32}, \Delta_{33}, \Delta_{41}, \Delta_{42}, \Delta_{43} \} \) (see Fig. 2.1). Then we construct a quadratic polynomial \( p_1(x, y, z) \in \text{span}\{1, \frac{x-x_0}{|\Delta_0|}^2, \frac{y-y_0}{|\Delta_0|}^2, \frac{z-z_0}{|\Delta_0|}^2, \frac{(x-x_0)(y-y_0)}{|\Delta_0|}^3, \frac{(x-x_0)(z-z_0)}{|\Delta_0|}^3, \frac{(y-y_0)(z-z_0)}{|\Delta_0|}^3, \frac{(z-z_0)^3}{|\Delta_0|}^3) \) on \( T_1 \) to obtain a third order approximation of conservative variable \( u \) and \( (x_0, y_0, z_0) \) is the barycenter of the target cell \( \Delta_0 \). Such quadratic polynomial has the same cell average of \( u \) on the target cell \( \Delta_0 \) and matches the cell averages of \( u \) on the other tetrahedrons in the set \( T_1 \setminus \{ \Delta_0 \} \) in a least square sense [19]:

\[
\frac{1}{|\Delta_0|} \int_{\Delta_0} p_1(x, y, z) dxdydz = \frac{1}{|\Delta_0|} \int_{\Delta_0} u(x, y, z, t) dxdydz = \bar{u}_0.
\]

(2.4)

and

\[
\min_{\Delta_{1\ell} \in T_1 \setminus \{ \Delta_0 \}} \left( \frac{1}{|\Delta_1\ell|} \int_{\Delta_1\ell} p_1(x, y, z) dxdydz - \bar{u}_\ell \right)^2.
\]

(2.5)

Remark 1. In order to determine ten degrees of freedom for \( p_1(x, y, z) \), we need to use ten distinct tetrahedral cells at least. But the quality of the mesh might be poor and some tetrahedral cells coincide. Such circumstance results in the lack of
Fig. 2.1. The geometrical graphs of small spatial stencils. From left to right and top to bottom: $T_2 = \{\Delta_0, \Delta_1, \Delta_{11}, \Delta_{12}, \Delta_{13}\}$, $T_3 = \{\Delta_0, \Delta_2, \Delta_{21}, \Delta_{22}, \Delta_{23}\}$, $T_4 = \{\Delta_0, \Delta_3, \Delta_{31}, \Delta_{32}, \Delta_{33}\}$, $T_5 = \{\Delta_0, \Delta_4, \Delta_{41}, \Delta_{42}, \Delta_{43}\}$, $T_6 = \{\Delta_0, \Delta_5, \Delta_{23}, \Delta_{43}\}$. And the big central stencil $T_1 = \bigcup_{k=2}^{6} T_r$.

enough ten distinct tetrahedral cells for reconstructing $p_1(x, y, z)$, we should search the next layer of neighboring cells and get enough cells for remedying such drawback.

Step 2. Select five small biased or central spatial stencils $T_2 = \{\Delta_0, \Delta_1, \Delta_{11}, \Delta_{12}, \Delta_{13}\}$, $T_3 = \{\Delta_0, \Delta_2, \Delta_{21}, \Delta_{22}, \Delta_{23}\}$, $T_4 = \{\Delta_0, \Delta_3, \Delta_{31}, \Delta_{32}, \Delta_{33}\}$, $T_5 = \{\Delta_0, \Delta_4, \Delta_{41}, \Delta_{42}, \Delta_{43}\}$ and $T_6 = \{\Delta_0, \Delta_5, \Delta_{23}, \Delta_{43}\}$, and reconstruct five linear polynomials $p_k(x, y) \in \text{span}\{1, \frac{x-x_0}{|\Delta_0|\frac{2}{3}}, \frac{y-y_0}{|\Delta_0|\frac{2}{3}}, \frac{z-z_0}{|\Delta_0|\frac{2}{3}}\}$, $k = 2, ..., 6$, such that each of them has the same cell average of $u$ on the target cell $\Delta_0$ and matches the cell averages of $u$ on the other tetrahedrons in a least square sense [19]:

$$
\frac{1}{|\Delta_0|} \int_{\Delta_0} p_k(x, y, z) dx dy dz = \bar{u}_0, \quad (2.6)
$$

and

$$
\min_{\Delta \in T_k \setminus \{\Delta_0\}} \left( \frac{1}{|\Delta_0|} \int_{\Delta_0} p_k(x, y, z) dx dy dz - \bar{u}_\ell \right)^2, \quad k = 2, ..., 6. \quad (2.7)
$$

Remark 2. We could also reconstruct sixteen three dimensional linear polynomials on the small stencils as in [38], where each stencil contains four cells and each linear polynomial has the same cell averages of $u$ on cells in the stencil, respectively. We find the fact that the method adopted in this paper is conciser.

Step 3. With the similar idea proposed by Levy, Puppo and Russo [22,23] for CWENO schemes, we rewrite $p_1(x, y, z)$ as

$$
p_1(x, y, z) = \gamma_1 \left( \frac{1}{\gamma_1} p_1(x, y, z) - \sum_{k=2}^{6} \frac{\gamma_k}{\gamma_1} p_k(x, y, z) \right) + \sum_{k=2}^{6} \gamma_k p_k(x, y, z). \quad (2.8)
$$

The summation of all terms at the right hand side of (2.8) equals $p_1(x, y, z)$ at anywhere on condition that $\gamma_1 \neq 0$. Since the quadratic polynomial $p_1(x, y, z)$ could obtain third order numerical approximation of $u(x, y, z, t)$ at any points and such unequal degree polynomials should confirm their convex combination relationship in spatial reconstructions of the WENO scheme, it is suitable to restrict all linear weights to be positive and their summation is one. Following the practice in [11,39,45], for example, one type of these linear weights is defined as $\gamma_1 = 0.95$ and $\gamma_2 = 0.01$, $l = 2, ..., 6$. We claim that such choice principle of the linear weights would not degrade the third order of accuracy with the similar methodologies adopted in [40–42]. However, some other robust WENO type schemes based on the artificial definition of the linear weights often degrade their optimal numerical accuracy.
Step 4. Compute the smoothness indicators $\beta_\ell$, $\ell = 1, \ldots, 6$, which measure how smooth the functions $p_\ell(x, y)$, $\ell = 1, \ldots, 6$, are in the target cell $\Delta_0$. The smaller these smoothness indicators are, the smoother the functions are in the target cell $\Delta_0$. We use the same recipe for the smoothness indicators [19,20]:

$$
\beta_\ell = \sum_{|l|=1}^{r} |\Delta_0|^{-\frac{2}{m-1}} \int_{\Delta_0} \left( \frac{\partial^{|l|}}{\partial x^l \partial y^l \partial z^l} p_\ell(x, y, z) \right)^2 \, dxdydz, \quad \ell = 1, \ldots, 6, \tag{2.9}
$$

where $l = (l_1, l_2, l_3)$, $|l| = l_1 + l_2 + l_3$. And for $\ell = 1$, $r$ equals 2; for $\ell = 2, \ldots, 6$, $r$ equals 1. Their expansions in Taylor series at the barycenter $(x_0, y_0, z_0)$ of the target cell $\Delta_0$ are

$$
\beta_1 = \sum_{|l|=1}^{1} \frac{\partial^{|l|}}{\partial x^l \partial y^l \partial z^l} u(x, y, z, t)|_{(x_0, y_0, z_0, t)}^2 |\Delta_0|^{\frac{2}{3}} (1 + O(\Delta_0^{\frac{2}{3}})) = O(\Delta_0^{\frac{2}{3}}), \tag{2.10}
$$

and

$$
\beta_\ell = \sum_{|l|=1}^{6} \frac{\partial^{|l|}}{\partial x^l \partial y^l \partial z^l} u(x, y, z, t)|_{(x_0, y_0, z_0, t)}^2 |\Delta_0|^{\frac{2}{3}} (1 + O(\Delta_0^{\frac{2}{3}})) = O(\Delta_0^{\frac{2}{3}}), \quad \ell = 2, \ldots, 6. \tag{2.11}
$$

Step 5. Compute the nonlinear weights based on the linear weights and the smoothness indicators. For instance, we use $\tau$ [40–42] which is simply defined as the absolute deference between $\beta_\ell$, $\ell = 1, \ldots, 6$ and is different to the formula specified in [3,8]. The difference expansions in Taylor series at $(x_0, y_0, z_0)$ are

$$
\beta_1 - \beta_\ell = O(\Delta_0^2), \quad \ell = 2, \ldots, 6. \tag{2.12}
$$

So it satisfies

$$
\tau = \left( \frac{\beta_1 - \beta_2 + |\beta_1 - \beta_3| + |\beta_1 - \beta_4| + |\beta_1 - \beta_5| + |\beta_1 - \beta_6|}{5} \right)^2 = O(\Delta_0^2). \tag{2.13}
$$

Then the associate nonlinear weights are defined as

$$
\omega_n = \frac{\bar{\omega}_n}{\sum_{\ell=1}^{6} \omega_\ell}, \quad \bar{\omega}_n = \gamma_n(1 + \frac{\tau}{\varepsilon + \beta_n}), \quad n = 1, \ldots, 6. \tag{2.14}
$$

Here $\varepsilon$ is a small positive number to avoid the denominator of (2.14) to become zero. By the implementation of (2.13) in the smooth region, they satisfy

$$
\frac{\tau}{\varepsilon + \beta_n} = O(\Delta_0^\frac{4}{3}), \quad n = 1, \ldots, 6, \tag{2.15}
$$

on condition that $\varepsilon \ll \beta_n$. Therefore, the nonlinear weights $\omega_n$, $n = 1, \ldots, 6$ satisfy the order accuracy condition $\omega_n = \gamma_n + O(\Delta_0^\frac{4}{3})$ [3,8], providing the third order accuracy to the WENO scheme narrated in [20,31]. We take $\varepsilon = 10^{-6}$ in all simulations in this paper.

Step 6. The new final reconstruction formulations of conservative values $u(x, y, z, t)$ at different quadrature points $(x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'})$, $\ell = 1, \ldots, 6$, $\ell\ell' = 1, 2, 3, 4$ on the boundaries of the target cell $\Delta_0$ are given by

$$
u^- (x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'}) \approx \omega_1 \left( \frac{1}{\gamma_1} p_1 (x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'}) - \sum_{\kappa=2}^{6} \frac{\gamma_\kappa}{\gamma_1} p_{\kappa} (x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'}) \right) + \sum_{\kappa=2}^{6} \omega_\kappa p_{\kappa} (x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'}) , \quad \ell = 1, \ldots, 6, \quad \ell\ell' = 1, 2, 3, 4. \tag{2.16}
$$

**Remark 3.** In each target cell $\Delta_0$, we reconstruct the third order accurate approximation values to $u(x, y, z, t)$ at twenty-four quadrature points on its four boundary triangular cells. In actually, we adopt the same linear weights and smoothness indicators for such points in this reconstruction procedure. We can obtain a reconstructed polynomial $Q(x, y, z) = \omega_1 \left( \frac{1}{\gamma_1} p_1 (x, y, z) - \sum_{\kappa=2}^{6} \frac{\gamma_\kappa}{\gamma_1} p_{\kappa} (x, y, z) \right) + \sum_{\kappa=2}^{6} \omega_\kappa p_{\kappa} (x, y, z)$ in the target tetrahedral cell $\Delta_0$ to approximate $u(x, y, z, t)$ at any arbitrary points inside the target cell $\Delta_0$, for example, $u^- (x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'}) \approx Q (x_G^{\ell, \ell'}, y_G^{\ell, \ell'}, z_G^{\ell, \ell'})$, $\ell = 1, \ldots, 6$, $\ell\ell' = 1, 2, 3, 4$. 


Step 7. Then the third order TVD Runge–Kutta time discretization method [32]

\[
\begin{align*}
    u^{(1)} &= u^n + \Delta t L(u^n), \\
    u^{(2)} &= \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t L(u^{(1)}), \\
    u^{n+1} &= \frac{1}{3} u^n + \frac{2}{3} u^{(2)} + \frac{1}{2} \Delta t L(u^{(2)}),
\end{align*}
\]

(2.17)
is used to solve (2.1). Finally, the fully discrete scheme both in space and time is designed on tetrahedral meshes.

Remark 4. We can apply two different methods to deal with the Euler equations. One method is to do such procedures in a component by component fashion. This way is easy to perform and cost less. But we would like to use a more cost expensive characteristic decomposition which is another method suggested by [19,26] for the application of high order schemes. So all of the reconstructions are performed in the local characteristic directions to avoid spurious oscillations for the Euler equations in this paper. We give a brief narration in the following. Let’s take one boundary triangular cell of the target tetrahedron which has an outward unit normal \(\vec{n} = (n_x, n_y, n_z)^T\). Let \(A\) be an average Jacobian matrix at one quadrature point:

\[
A = n_x \frac{\partial f}{\partial u} + n_y \frac{\partial g}{\partial u} + n_z \frac{\partial r}{\partial u}.
\]

(2.18)
The Roe’s mean matrix [28] is used for three dimensional Euler equations. We define \(R\) as the matrix of right eigenvectors and \(L\) as the matrix of left eigenvectors of \(A\). So the scalar third order tetrahedral WENO scheme can be directly applied to each of the five characteristic fields, i.e., to each component of the vector \(v = Lu\). With the reconstructed point values \(v\), we denote the reconstructed point values \(u\) as \(u = Rv\).

As pointed out in [25,30,38], if the quadrature points are not chosen properly or the geometry of the computational meshes is rigid, some linear weights are negative or even do not exist at all, then we have to add some procedures to deal with such drawback [30]. In this paper, we use one quadratic polynomial and five linear polynomials instead of one quadratic polynomial and sixteen linear polynomials in [38], artificially choose positive linear weights with a minor restriction and apply a new formula of nonlinear weights on tetrahedral meshes. By doing so, the scheme can maintain the third order accuracy in smooth region and avoid spurious oscillations adjacent to the discontinuities. Comparing with some WENO schemes proposed on tetrahedral meshes, we can find such new WENO scheme is simple, efficient and robust for solving hyperbolic conservation laws.

3. Numerical results

In this section we provide some numerical results to demonstrate the performance of the new third order finite volume WENO scheme with six unequal size spatial stencils on tetrahedral meshes described in section 2. For the purpose of evaluating whether the random choice of the positive linear weights would sustain the third order accuracy in smooth region or not, we set four different types of linear weights in associated numerical accuracy test cases as: (1) \(\gamma_1 = 0.95\) and \(\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0.01\); (2) \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 1.0/6.0\); (3) \(\gamma_1 = 0.01\) and \(\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0.198\); (4) \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0.01\) and \(\gamma_6 = 0.95\). After that, we would like to recover \(\gamma_1 = 0.95, \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0.01\) in the other test cases, unless specified otherwise.

Example 3.1. We solve the following linear scalar equation

\[
u_t + u_x + u_y + u_z = 0,
\]

(3.1)

with computing field \([-2, 2] \times [-2, 2] \times [-2, 2]\) on uniform tetrahedral meshes. The initial condition is \(u(x, y, z, 0) = \sin(\pi (x + y + z))/2\) and periodic boundary conditions are applied in each direction. The final time is \(t = 1\). The errors and numerical orders of accuracy for the third order finite volume WENO scheme with different types of linear weights are shown in Table 3.1. We can see that the new WENO scheme keeps the designed order of accuracy, however the magnitude of the absolute truncation errors is different.

Example 3.2. We solve the following Burgers’ equation

\[
u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y + \left(\frac{u^2}{2}\right)_z = 0,
\]

(3.2)

with computing field \([-3, 3] \times [-3, 3] \times [-3, 3]\) on the uniform tetrahedral meshes. The initial condition is \(u(x, y, z, 0) = 0.5 + \sin(\pi (x + y + z))/3\) and periodic boundary conditions are applied in each direction. The final time is \(t = 0.5/\pi^2\). The errors and numerical orders of accuracy for the new WENO scheme with four different types of linear weights are shown in Table 3.2. We can also see that the new scheme keeps the designed third order without degrading, and the magnitude of the absolute truncation errors are different for different linear weights on the same mesh levels.
Table 3.1

\(u_t + u_x + u_y + u_z = 0\). Periodic boundary conditions in each direction. \(T = 1\). \(L^1\) and \(L^\infty\) errors. Uniform tetrahedral mesh.

<table>
<thead>
<tr>
<th>Tetrahedrons</th>
<th>WENO scheme (1)</th>
<th>WENO scheme (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(L^1) error</td>
<td>(L^\infty) error</td>
</tr>
<tr>
<td>750</td>
<td>2.87E-1</td>
<td>4.65E-1</td>
</tr>
<tr>
<td>6000</td>
<td>4.53E-2</td>
<td>7.84E-2</td>
</tr>
<tr>
<td>20250</td>
<td>1.35E-2</td>
<td>2.95E-2</td>
</tr>
<tr>
<td>48000</td>
<td>5.83E-3</td>
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</tr>
<tr>
<td>93750</td>
<td>3.01E-3</td>
<td>4.87E-3</td>
</tr>
</tbody>
</table>

WENO scheme (3)

<table>
<thead>
<tr>
<th>Tetrahedrons</th>
<th>WENO scheme (3)</th>
<th>WENO scheme (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(L^1) error</td>
<td>(L^\infty) error</td>
</tr>
<tr>
<td>750</td>
<td>3.96E-1</td>
<td>6.48E-1</td>
</tr>
<tr>
<td>6000</td>
<td>6.26E-2</td>
<td>1.36E-2</td>
</tr>
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<td>20250</td>
<td>1.35E-2</td>
<td>3.76E-2</td>
</tr>
<tr>
<td>48000</td>
<td>5.74E-3</td>
<td>1.36E-2</td>
</tr>
<tr>
<td>93750</td>
<td>2.99E-3</td>
<td>9.58E-3</td>
</tr>
</tbody>
</table>

Remark 5. The new finite volume WENO scheme could obtain third order accuracy in smooth region with different type of linear weights for such accuracy test cases. We find the WENO scheme (1) and WENO scheme (4) could get less \(L^1\) and \(L^\infty\) truncation errors in comparison with that of the WENO schemes (2) and (3). Hereafter, we only apply the WENO scheme (1) for simulating the latter examples for simplicity.

Example 3.4. We solve the one dimensional Euler equations

\[ \rho \frac{\partial}{\partial t} \left( \frac{u}{E} \right) + \rho \frac{\partial}{\partial x} \left( u^2 + p \right) = 0. \tag{3.4} \]
Table 3.3
3D-Euler equations: initial data \( \rho(x, y, z, 0) = 1 + 0.2 \sin(\pi(x + y + z)/3) \), \( u(x, y, z, 0) = 1 \), \( v(x, y, z, 0) = 1 \), \( w(x, y, z, 0) = 1 \) and \( p(x, y, z, 0) = 1 \). Periodic boundary conditions in each direction. \( T = 1 \). \( L^1 \) and \( L^\infty \) errors. Uniform tetrahedral mesh.

<table>
<thead>
<tr>
<th>Tetrahedrons</th>
<th>WENO scheme (1)</th>
<th>WENO scheme (2)</th>
<th>WENO scheme (3)</th>
<th>WENO scheme (4)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>( L^1 ) error</td>
<td>order</td>
<td>( L^\infty ) error</td>
<td>order</td>
</tr>
<tr>
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<td></td>
<td>1.04E-1</td>
<td></td>
</tr>
<tr>
<td>6000</td>
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<tr>
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<td>2.82</td>
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<td>2.87</td>
</tr>
<tr>
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<td>9.27E-4</td>
<td>2.96</td>
<td>1.47E-3</td>
<td>2.93</td>
</tr>
</tbody>
</table>

Fig. 3.1. The Lax problem. \( T = 0.16 \). From left to right: density; density zoomed in; 3D density surface. Solid line: the exact solution; squares: the results of WENO scheme. The mesh points on the boundary are uniformly distributed with cell length \( \Delta x = \Delta y = \Delta z = 1/100 \).

Here \( \rho \) is density, \( u \) is the velocity in \( x \) direction, \( E \) is total energy and \( p \) is pressure. The Riemann initial condition for the Lax problem:

\[
(\rho, u, p, \gamma)^T = \begin{cases} 
(0.445, 0.698, 3.528, 1.4)^T, & x \in [-0.5, 0), \\
(0.5, 0, 0.571, 1.4)^T, & x \in [0, 0.5].
\end{cases}
\] (3.5)

This three dimensional tetrahedral WENO scheme is applied to the one dimensional shock tube problem. The solution of the Euler equations (3.4) lies in the domain of \([-0.5, 0.5] \times [-z_0, z_0] \times [-z_0, z_0] \) \( \times [z_0, \Delta x] \) with a tetrahedralization of 101 vertices in the \( x \) direction and 5 vertices in the \( y \) and \( z \) directions, respectively. The velocities in the \( y \) and \( z \) directions are set as 0 and periodic boundary conditions are applied in the \( y \) and \( z \) directions. The final time is \( t = 0.16 \).

We present the exact solution and the computed density \( \rho \) obtained with the new third order finite volume WENO scheme in Fig. 3.1. We observe that the computational results obtained by the new WENO scheme are good.

**Example 3.5.** We solve the one dimensional Euler equations (3.4) with Riemann initial condition for the Sod problem:

\[
(\rho, u, p, \gamma)^T = \begin{cases} 
(1, 0, 2.5, 1.4)^T, & x \in [-5, 0), \\
(0.125, 0, 0.25, 1.4)^T, & x \in [0, 5].
\end{cases}
\] (3.6)

The solution of the Euler equations (3.4) lies in the domain of \([-5, 5] \times [-2\Delta y, 2\Delta y] \times [-2\Delta z, 2\Delta z] \) \( \times [\Delta x = \Delta y = \Delta z] \) with a tetrahedralization of 101 vertices in the \( x \) direction and 5 vertices in the \( y \) and \( z \) directions, respectively. The velocities in the \( y \) and \( z \) directions are set as 0 and periodic boundary conditions are applied in the \( y \) and \( z \) directions. The final time is \( t = 2 \). We present the exact solution and the computed density \( \rho \) obtained with the new third order finite volume WENO scheme in Fig. 3.2. The numerical results computed by the WENO scheme are good for this one dimensional test example.
The difference density is solved by the WENO scheme. The mesh points on the boundary are uniformly distributed with cell length $\Delta x = \Delta y = \Delta z = 1/100$.

Example 3.6. A higher order scheme would show its advantage when the solution contains both shocks and complex smooth region structures. A typical example for this is the problem of shock interaction with entropy waves [31]. We solve the Euler equations (3.4) with a moving Mach = 3 shock interacting with sine waves in density: $(\rho, u, p, \gamma)^T = (3.857143, 2.629369, 10.333333, 1.4)^T$ for $x \in [-5, -4]$; $(\rho, u, p, \gamma)^T = (1 + 0.2 \sin(5x), 0, 1, 1.4)^T$ for $x \in [-4, 5]$. The solution of the Euler equations (3.4) lies in the domain of $[-5, 5] \times [-2\Delta y, 2\Delta y] \times [-2\Delta z, 2\Delta z]$ ($\Delta x = \Delta y = \Delta z$) with a tetrahedralization of 401 vertices in the $x$ direction and 5 vertices in the $y$ and $z$ directions, respectively. The velocities in the $y$ and $z$ directions are set as 0 and periodic boundary conditions are applied in the $y$ and $z$ directions. The computed density $\rho$ is plotted at $t = 1.8$ against the referenced “exact” solution which is a converged solution computed by the finite difference fifth order WENO scheme [20] with 2000 grid cells in Fig. 3.3. The new type of third order finite volume WENO scheme could get good resolution for this benchmark example.

Example 3.7. We now consider the interaction of two blast waves. The initial conditions are

$$
(\rho, u, p, \gamma)^T = \begin{cases} 
(1, 0, 10^2, 1.4)^T, & 0 < x < 0.1, \\
(1, 0, 10^{-2}, 1.4)^T, & 0.1 < x < 0.9, \\
(1, 0, 10^2, 1.4)^T, & 0.9 < x < 1.
\end{cases} 
$$

The solution of the Euler equations (3.4) lies in the domain of $[0, 1] \times [-2\Delta y, 2\Delta y] \times [-2\Delta z, 2\Delta z]$ ($\Delta x = \Delta y = \Delta z$) with a tetrahedralization of 401 vertices in the $x$ direction and 5 vertices in the $y$ and $z$ directions, respectively. The velocities in the $y$ and $z$ directions are set as 0, and periodic boundary conditions are applied in the $y$ and $z$ directions. The computed density $\rho$ is plotted at $t = 0.038$ against the reference “exact” solution which is a converged solution computed by the finite difference fifth order WENO scheme [20] with 2000 grid cells in Fig. 3.4. The new WENO scheme could get good performance as usual.
Example 3.8. We solve the same nonlinear Burgers’ equation (3.2) with the same initial condition $u(x, y, z, 0) = 0.5 + \sin(\pi(x + y + z)/3)$, except that we plot the results at $t = 5/\pi^2$ when a shock has already appeared in the solution. We show the contours on the surface and one dimensional cutting-plot along $x = y, z = 0$ of the solutions by the third order finite volume WENO scheme in Fig. 3.5. We can see that the scheme gives non-oscillatory shock transitions for this scalar problem.

Example 3.9. This case concerns the transonic flow over the Onera M6 wing [14]. This problem is a classic CFD validation case for external flows because of its simple geometry combined with complexities of transonic flow. This benchmark test case uses the following flow conditions: $M_{\infty} = 0.84$ and angle of attack $\alpha = 30.06^\circ$. The computational domain is set as $\sqrt{x^2 + y^2 + z^2} \leq 16$ and $z \geq 0$, which consists of 143645 tetrahedrons and 24382 points with 1311 triangles over the surface. The surface mesh and Mach number contours are shown in Fig. 3.6. And then the reduction of density residual as a function of the number of iterations is shown in the same Fig. 3.6. We can see that the new scheme performs well in this test case.

Example 3.10. We use INRIA’s 3D tetrahedral elements for the BTC0 (streamlined body, laminar) test case in project ADIGMA [27] with the initial conditions: $M_{\infty} = 0.5$ and angle of attack $\alpha = 0^\circ$. The computational domain is $\sqrt{x^2 + y^2 + z^2} \leq 10$ which consists of 191753 tetrahedrons and 33708 points with 8244 triangles over the surface. The surface mesh is shown in Fig. 3.7. We give Mach number and pressure in Fig. 3.7. And then the reduction of density residual as a function of the number of iterations is shown in Fig. 3.7. It shows that the scheme gives good resolution.

Example 3.11. We consider the inviscid Euler transonic flow past a Russian jet plane whose surface mesh is also modeled by INRIA. The initial conditions are $M_{\infty} = 0.85$ and angle of attack $\alpha = 1^\circ$. The computational domain is $\sqrt{x^2 + y^2 + z^2} \leq 100$ which consists of 88169 tetrahedrons and 22663 points with 33092 triangles over the surface. The surface mesh is shown
Fig. 3.6. Onera M6 wing problem. $M_\infty = 0.84$, angle of attack $\alpha = 3.06^\circ$. From left to right and top to bottom: Onera M6 wing surface mesh, zoomed in; Mach number contours plot on the surface; the reduction of density residual as a function of the number of iterations.

in Fig. 3.8. Mach number and pressure are shown in 3.8. And then the reduction of density residual as a function of the number of iterations is shown in Fig. 3.8. We can see that the new finite volume WENO scheme performs well for this subsonic problem.

4. Concluding remarks

We investigate designing a new third order finite volume WENO scheme on tetrahedral meshes. Generally speaking, we do the innovations in such aspects: we only select six unequal size spatial stencils [22,23,43,44] and reconstruct six unequal degree polynomials based on them in comparison with the selection of seventeen spatial stencils and the associated seventeen polynomials in [38] for the spatial reconstructions on tetrahedral meshes; it is a first time to take effort in compounding two different type of WENO schemes without introducing any switching mechanism by adjusting some threshold parameters [25]; it is a new way of bypassing the calculation of the optimal linear weights for high order accuracy and it avoids the performance of dealing with the negative linear weights [30]; it takes a new formulation of the nonlinear weights for unequal spatial stencils on tetrahedral meshes. Such spatial reconstruction framework of the third order approximation at any quadrature points by one quadratic polynomial defined on big central spatial stencil and five linear polynomials defined on small biased or central stencils is easily implemented to fourth and higher order finite volume WENO schemes. Such WENO schemes only depend on the usage of suitable high degree polynomial for obtaining high order approximation in smooth region and can switch to at least one of similar five linear polynomials for keeping essentially non-oscillatory property in nonsmooth region. So it is a very simple way to get different finite volume WENO schemes via using the same framework on tetrahedral meshes. And it is a new and promising method that could be easily implemented to a wide variety of fields, such as anisotropic moving mesh simulations, ALE methods, adaptive mesh calculations, hybrid mesh simulations and so on.
Fig. 3.7. BTC0 problem. $M_\infty = 0.5$, angle of attack $\alpha = 0^\circ$. From left to right and top to bottom: BTC0 surface mesh, zoomed in; Mach number contours plot on the surface; pressure contours plot on the surface; the reduction of density residual as a function of the number of iterations.

Fig. 3.8. Russian jet plane problem. $M_\infty = 0.85$, angle of attack $\alpha = 1^\circ$. From left to right and top to bottom: Russian jet plane surface mesh, zoomed in; Mach number contours plot on the surface; pressure contours plot on the surface; the reduction of density residual as a function of the number of iterations.
References