SSP IMEX Runge-Kutta WENO scheme for generalized Rosenau-KdV-RLW equation

Muyassar Ahmat and Jianxian Qiu

Abstract

In this paper, we present a third-order weighted essentially non-oscillatory (WENO) method for generalized Rosenau-KdV-RLW equation. The third order finite difference WENO reconstruction and central finite differences are applied to discrete advection terms and other terms, respectively, in spatial discretization. In order to achieve the third order accuracy both in space and time, four stage third-order L-stable SSP Implicit-Explicit Runge-Kutta method (Third-order SSP EXRK method and third-order DIRK method) is applied to temporal discretization. The high order accuracy and essentially non-oscillatory property of finite difference WENO reconstruction are shown for solitary wave and shock wave for Rosenau-KdV and Rosenau-KdV-RLW equations. The efficiency, reliability and excellent SSP property of the numerical scheme are demonstrated by several numerical experiments with large CFL number.

Key Words: Rosenau-KdV-RLW equation, WENO reconstruction, finite difference method, SSP implicit-explicit Runge-Kutta method.

AMS(MOS) subject classification: 65M60, 35L65

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1 Introduction

The dispersive shallow water wave dynamics is one of the active scientific research areas during the past several decades, which usually describes the wave behavior at lake shores and beaches. Among all these types of models, the Rosenau equation [1] describes the wave-wave and wave-wall phenomenon of the dense discrete system. In order to further consider the nonlinear wave behavior on shallow water dynamics, the viscous term $u_{xxx}$ or $u_{xxt}$ need to be included in this equation, which leads to the achievement of Rosenau-RLW equation:

$$u_t + \alpha u_x + \delta u_{xxt} + \nu u_{xxxx} + \varepsilon (u^p)_x = 0. \quad (1.1)$$

or Rosenau-KdV equation:

$$u_t + \nu u_{xxxx} + \alpha u_x + \theta u_{xxx} + \varepsilon (u^p)_x = 0. \quad (1.2)$$

There have been many difficulties in evaluating analytical solutions of nonlinear dispersive wave equations and so on the development of numerical schemes. Even so, one derived the solitary wave solution and singular soliton solution for the Rosenau-KdV equation by the ansatz method as well as the semi-inverse variational principle [2] while the shock solution of this equation was given by Ebadi [3].

Significant numerical studies have been done on the Rosenau-KdV equation [4, 5]. Two-level nonlinear implicit Crank-Nicolson difference scheme and three-level linear-implicit difference scheme were presented to solve two-dimensional generalized Rosenau-KdV equation by Atouani [4]. Their experiment proved that both schemes were uniquely solvable, unconditionally stable and second-order convergent in $L_1$ norm, the linearized scheme was more effective in terms of accuracy and computational cost. Wang and Dai [5] proposed a conservative unconditionally stable finite difference scheme with $o(h^4 + \tau^2)$ for the generalized Rosenau-KdV equation in both one and two dimension, where $h$ is spatial step and $\tau$ is temporal step, respectively.

A mass-preserving scheme which combined a high-order compact scheme and a three-level average difference iterative algorithm was analyzed and tested for the Rosenau-RLW
equation in [6]. In their work, they focused on the development of the approach for solving the nonlinear implicit scheme in aim to improve the accuracy of approximate solutions. The Rosenau-RLW equation was also solved by second-order nonlinear finite element Galerkin-Crank-Nicolson method which was linearized by predictor-correction extrapolation technique in [7]. An energy conservative two-level fourth-order nonlinear implicit compact difference scheme for three dimensional Rosenau-RLW equation was designed by Li [8] and an iterative algorithm was introduced to generate this nonlinear algebraical system.

In this paper, we focus on one-dimensional generalized Rosenau-KdV-RLW equation. In order to keep this model in a generalized setting, the Rosenau-KdV-RLW equation is written as:

$$u_t + \delta u_{xxx} + \nu u_{xxxx} + \alpha u_x + \theta u_{xxx} + \varepsilon (u^p)_x = 0.$$  

where $u(x, t)$ denote the profile of the wave while $x$ and $t$ are the spatial and temporal variables, respectively. $\alpha > 0, \varepsilon > 0$ are the parameters of linear and nonlinear advection terms, $p \geq 2$ is the parameter of power law nonlinearity. $\theta, \delta, \nu$ are the parameters of KdV, RLW, Rosenau terms, respectively.

Rosenau-KdV-RLW equation has been studied both theoretically and numerically in recent years. Ansatz approach and semi-inverse variational principle were used to determine the singular and shock solution, and the conservation laws of the Rosenau-KdV-RLW equation with power law nonlinearity were computed by the aid of multiplier approach in Lie symmetry analysis in [9] and [10]. A three-level second-order accurate weighted average implicit finite difference scheme was presented by Wongsaijai [11] to solve the Rosenau-KdV-RLW equation. Wang [12] introduced a three-level linear conservative implicit finite difference scheme for solving the Rosenau-KdV-RLW equation which was easy to implement and had simple computational structure. A multi-symplectic scheme and an energy-preserving scheme based on the multi-symplectic Hamiltonian formulation of the equation were tested for the generalized Rosenau-type equation in [13]. These methods were implemented efficiently by the discrete fast Fourier transform with spectral accuracy in space while
second-order accuracy in time.

The Implicit-Explicit (IMEX) Runge-Kutta method is an effective time solver with the advantages of loosening the CFL restriction caused by the Explicit scheme and reducing the computational cost caused by Implicit method reasonably for PDEs which contains stiff and non-stiff terms all together, and applied generally for this type of PDEs [15–17]. In order to ensure the stability stands for this type of large ODE system obtained from spatial discretization, It is much safer to use IMEX Runge-Kutta methods with strong stability preserving (SSP) properties [18–20].

The weighted essentially non-oscillatory (WENO) method is mostly applied for hyperbolic conservation laws with the advantages of the capability to achieve high-order accuracy in smooth regions while maintaining stable, non-oscillatory property in sharp or stiff region [21–23]. Here, we use the same approach for the solitary wave solution and shock wave solution of Rosenau-KdV-RLW equation.

In this paper, we use SSP IMEX Runge-Kutta method in time direction [20] and third-order finite difference WENO reconstruction [22] for advection terms, and central finite difference for remaining terms in spatial direction for (1.3). The advantages of finite difference WENO reconstruction is exploited in wave motions, especially shock wave. To be specific, we use third-order finite difference WENO scheme for advection terms which are treated explicitly in the time direction. The rest of (1.3) is treated by high order central finite difference method in space and treated implicitly in time.

The paper is arranged as follows. In Section 2, the third-order finite difference WENO scheme and high order finite difference method are performed. In Section 3, the third-order SSP IMEX Runge-Kutta scheme is given for the treatment in time. Extensive numerical results are proposed in Section 4 to illustrate the accuracy and efficiency of the present method. Concluding remarks are given in the final section.
2 Spatial discretization

We use a uniform mesh of cell size $h$ in space. The uniform mesh is distributed as follows:

$$x \in [x_l, x_r], \quad x_i = x_l + i h, \quad i = 1 : N - 1, \quad x_1 = x_l, \quad x_N = x_r,$$

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \quad x_{i+\frac{1}{2}} = \frac{x_{i+1} + x_i}{2}.$$  

We will give a brief sketch of the algorithms about third order finite difference WENO scheme with Lax-Friedrichs flux splitting which is used to treat $f(u) = au + \varepsilon u^p$, here $f(u)_x$ can be reformulated as

$$f(u)_x|_{x = x_i} \approx \frac{1}{h} (\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}). \quad (2.1)$$

where $\hat{f}_{i+\frac{1}{2}}, \hat{f}_{i-\frac{1}{2}}$ are the numerical fluxes such that right hand side of (2.1) is a third order approximation to $f(u)_x|_{x = x_i}$. $u_i(t)$ is defined as a nodal point value $u(x_i, t)$.

In finite difference WENO reconstruction, flux splitting has to be done for the purpose of stability. For flux $f(u)$, we perform the "Lax-Friedrichs flux splitting":

$$f^+(u) = \frac{1}{2} (f(u) + \alpha u), \quad f^-(u) = \frac{1}{2} (f(u) - \alpha u). \quad (2.2)$$

where $\alpha = \max_u |f'(u)|$, so that

$$f(u) = f^-(u) + f^+(u). \quad (2.3)$$

satisfying

$$\frac{d}{du} f^+(u) \geq 0, \quad \frac{d}{du} f^-(u) \leq 0. \quad (2.4)$$

In here, we only recall the reconstruction of $f^+(u)$ at point $x_{i+\frac{1}{2}}$. We choose big stencil $\Gamma = [I_{i-1}, I_i, I_{i+1}]$. In this stencil, we can obtain a second degree polynomial $H(x)$ which is based on the nodal point information of the flux splitting and satisfying:

$$\frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} H(x) dx = f_j^+, \quad j = i - 1, i, i + 1 \Rightarrow H(x_{i+\frac{1}{2}}) = -\frac{1}{6} f_{i-1}^+ + \frac{5}{6} f_i^+ + \frac{1}{3} f_{i+1}^+. \quad (2.5)$$

In two small stencil $\Gamma_1 = [I_{i-1}, I_i]$, $\Gamma_2 = [I_i, I_{i+1}]$, we can obtain two linear polynomial $H_1(x), H_2(x)$ respectively.

$$\frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} H_1(x) dx = f_j^+, \quad j = i - 1, i \Rightarrow H_1(x_{i+\frac{1}{2}}) = -\frac{1}{2} f_{i-1}^+ + \frac{3}{2} f_i^+. \quad (2.6)$$
\[
\frac{1}{h} \int_{x_j - \frac{h}{2}}^{x_j + \frac{h}{2}} H_2(x) \, dx = f_j^+, \quad j = i, i + 1 \Rightarrow H_2(x_{i+\frac{1}{2}}) = \frac{1}{2}f_i^+ + \frac{1}{2}f_{i+1}^+. \quad (2.7)
\]

Define the linear weights \( r_1, r_2 \), such that
\[
H(x_{i+\frac{1}{2}}) = r_1 H_1(x_{i+\frac{1}{2}}) + r_2 H_2(x_{i+\frac{1}{2}}). \quad (2.8)
\]

We have \( r_1 = \frac{1}{3}, r_2 = \frac{2}{3} \). The smoothness indicators to measure the smoothness of \( H_1(x) \) and \( H_2(x) \) are defined as
\[
\beta_1 = (f_i^+ - f_{i-1}^+)^2, \quad \beta_2 = (f_{i+1}^+ - f_i^+)^2. \quad (2.9)
\]

then we define the nonlinear weights as follows:
\[
w_j = \frac{\tilde{w}_j}{w_1 + w_2}, \quad \tilde{w}_j = \frac{r_j}{(\epsilon + \beta_j)^2}, \quad j = 1, 2.
\]

where \( \epsilon \) is very small and positive, which is chosen to avoid the denominator becoming 0 and typically chosen to be \( \epsilon = 10^{-6} \) in the calculation.

Finally, we obtain the third-order approximation:
\[
\hat{f}_{i+\frac{1}{2}}^+ = w_1 H_1(x_{i+\frac{1}{2}}) + w_2 H_2(x_{i+\frac{1}{2}}). \quad (2.10)
\]

\( \hat{f}_{i+\frac{1}{2}}^- \) also can be obtained in the same way, so that
\[
\hat{f}_{i+\frac{1}{2}} = \hat{f}_{i+\frac{1}{2}}^+ + \hat{f}_{i+\frac{1}{2}}^- \quad (2.11)
\]

For the other high order derivation terms in (1.3), we simply use high order finite difference to approximate them.

\[
(u_i)_{xx} = -\frac{1}{12} u_{i+2} + \frac{4}{3} u_{i+1} - \frac{5}{7} u_i + \frac{4}{3} u_{i-1} - \frac{1}{12} u_{i-2} = \left( \frac{\partial^2 u}{\partial x^2} \right)_i - \frac{1}{90} \left( \frac{\partial^6 u}{\partial x^6} \right)_i h^4 + \cdots \quad (2.12)
\]

\[
(u_i)_{xxx} = -\frac{1}{8} u_{i+3} + u_{i+2} - \frac{13}{8} u_{i+1} + \frac{13}{7} u_{i-1} - u_{i-2} + \frac{1}{8} u_{i-3} = \left( \frac{\partial^3 u}{\partial x^3} \right)_i - \frac{7}{120} \left( \frac{\partial^7 u}{\partial x^7} \right)_i h^4 + \cdots \quad (2.13)
\]

\[
(u_i)_{xxxx} = -\frac{1}{6} u_{i+4} + 2 u_{i+3} - \frac{13}{2} u_{i+2} + \frac{23}{3} u_i - \frac{13}{7} u_{i-1} + 2 u_{i-2} - \frac{1}{6} u_{i-3} = \left( \frac{\partial^4 u}{\partial x^4} \right)_i - \frac{7}{240} \left( \frac{\partial^8 u}{\partial x^8} \right)_i h^4 + \cdots \quad (2.14)
\]
3 The third order SSP IMEX Runge-Kutta method

We now rewrite (1.3) as:

\[(u + \delta u_{xx} + \nu u_{xxxx})_t = -\alpha u_x - \theta u_{xxx} - \varepsilon (u^p)_x.\]  

and obtain the semi-discrete form by discretizing (3.1) at \((x_i, t)\):

\[\left(u_i + \delta (u_i)_{xx} + \nu (u_i)_{xxxx}\right)_t = -\alpha (u_i)_x - \varepsilon \left((u_i)^p\right)_x - \theta (u_i)_{xxx}.\]  

We use the above third-order finite difference WENO reconstruction and finite difference discretization procedure for (3.2):

\[
\frac{d}{dt}\left[u_i + \delta \frac{-1}{12} u_{i+2} + \frac{4}{3} u_{i+1} - \frac{5}{2} u_i + \frac{4}{3} u_{i-1} - \frac{1}{12} u_{i-2}
+ \nu \frac{-1}{6} u_{i+3} + 2 u_{i+2} - \frac{13}{2} u_{i+1} + \frac{28}{3} u_i - \frac{13}{2} u_{i-1} + 2 u_{i-2} - \frac{1}{6} u_{i-3}}{h^2}
\right] = L\left(u(x_i, t)\right) + \left[-\theta \frac{-1}{8} u_{i+3} + u_{i+2} - \frac{13}{8} u_{i+1} + \frac{13}{8} u_{i-1} - u_{i-2} + \frac{1}{8} u_{i-3}\right].
\]

and write the whole system in matrix form:

\[
\frac{d}{dt}[AU] = L(U) + BU.
\]  

where \(L(u)\) is the high order spatial discrete formulation of \(-f_x(u)\) obtained from WENO reconstruction. Finally, we obtain the matrix form of the semi-discrete system (1.3):

\[
\frac{dU}{dt} = A^{-1} L(U) + A^{-1} BU.
\]  

where

\[
A = \begin{pmatrix}
1 & -\frac{5\delta}{2h^2} + \frac{28\nu}{3h^4} & \frac{4\delta}{3h^4} - \frac{13\nu}{2h^4} & -\frac{\delta}{12h^2} + \frac{2\nu}{h^3} & -\frac{\nu}{6h^4} \\
-\frac{\delta}{12h^2} & -\frac{\delta}{12h^2} + \frac{2\nu}{h^3} & -\frac{\nu}{6h^4} & & \\
-\frac{\nu}{6h^4} & & & & \\
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1 & -\frac{5\delta}{2h^2} + \frac{28\nu}{3h^4} & \frac{4\delta}{3h^4} - \frac{13\nu}{2h^4} & -\frac{\delta}{12h^2} + \frac{2\nu}{h^3} & -\frac{\nu}{6h^4} \\
\end{pmatrix}_{N \times N}.
\]
Explicit term for stiff parts caused by high order spatial derivation. This method has the following
by third-order WENO reconstruction, and L-stable diagonally implicit Runge-Kutta (DIRK)
ically S-stage SSP Explicit Runge-Kutta method for advection terms which are computed
behavior, so we use inflow and outflow boundary at n-th time level and add these ghost cell
values into the first and last column of matrix B.

Further, we will use S-stage SSP IMEX-RK scheme for the temporal discretization, specifically S-stage SSP Explicit Runge-Kutta method for advection terms which are computed by third-order WENO reconstruction, and L-stable diagonally implicit Runge-Kutta (DIRK) method for stiff parts caused by high order spatial derivation. This method has the following form when it is applied to (3.5):

\[
U^{(m)} = U^n + \tau \sum_{q=1}^{m-1} \tilde{a}_{mq} A^{-1} L(U^{(q)}) + \tau \sum_{q=1}^{S} \hat{a}_{mq} A^{-1} BU^{(q)},
\]

\[
U^{n+1} = U^n + \tau \sum_{q=1}^{S} \hat{b}_q A^{-1} L(U^{(q)}) + \tau \sum_{q=1}^{S} \hat{b}_q A^{-1} BU^{(q)}.
\]

with the double Butchart tableau

\[
\begin{array}{c|c|c|c|c}
\hat{c} & \tilde{A} & \hat{c} & \hat{A} \\
\hline
\hat{b} & \hat{b} & \hat{b}
\end{array}
\]

Explicit & Implicit

where \( \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_S)^T \), \( \hat{c} = (\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_S)^T \), \( \hat{b} = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_S)^T \), \( \hat{c} = (\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_S)^T \)
are coefficient vectors and \( \tilde{A} = (\tilde{a}_{mq}), \hat{a}_{mq} = 0 \) for \( q \geq m \) and \( \hat{A} = (\hat{a}_{mq}) \) are \( S \times S \) matrices.

By using the above procedure, we present the algorithm for numerical solution.

• Explicit term

\[
U_s^{(m)} = U^n + \tau \sum_{q=1}^{m-2} \tilde{a}_{mq} A^{-1} L(U^{(q)}) + \tau \tilde{a}_{m,m-1} A^{-1} L(U^{(m-1)}),
\]

\[
B = \begin{pmatrix}
-\frac{139}{8h^3} + \frac{1}{h^3} & \frac{1}{h^3} & \cdots & \cdots & 0 \\
\frac{1}{h^3} & -\frac{139}{8h^3} + \frac{1}{h^3} & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \frac{1}{h^3} \\
\frac{1}{h^3} & \cdots & \cdots & \cdots & -\frac{139}{8h^3} + \frac{1}{h^3}
\end{pmatrix}_{N \times N}
\]
• Implicit term
\[
U^{(m)} = U_*^{(m)} + \tau \sum_{q=1}^{m-1} \hat{a}_{mq} A^{-1} BU^{(q)} + \tau \hat{a}_{mn} A^{-1} BU^{(m)},
\]

• Final solution at next time level
\[
U^{n+1} = U^n + \tau \sum_{q=1}^{S} \hat{b}_q A^{-1} L(U^{(q)}) + \tau \sum_{q=1}^{S} \hat{b}_q A^{-1} BU^{(q)}.
\]

In this paper, we apply the third-order L-stable SSP IMEX Runge-Kutta method with the Butchar tableau given as in [20]:

\[
\begin{pmatrix}
\hat{c} & b \\
\hat{c} & \hat{A}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 0
0 & 1 & 1 & 0 & 0
0 & 0 & 1 & 1 & 0
0 & 0 & 1 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
\hat{c} & A \\
\hat{c} & \hat{A}
\end{pmatrix} = \begin{pmatrix}
\hat{\alpha} & \hat{\alpha} & 0 & 0 & 0
\hat{\alpha} & \hat{\alpha} & \hat{\alpha} & 0 & 0
0 & -\hat{\alpha} & \hat{\alpha} & 0 & 0
1 & 0 & 1 - \hat{\alpha} & \hat{\alpha} & 0
1 & \hat{\gamma} & -\hat{\gamma} - \hat{\alpha} & \hat{\alpha} & 0
0 & 1 & 0 & 0 & 0
0 & \frac{1}{6} & 0 & 0 & 0
\end{pmatrix}.
\]

with $\hat{\alpha} = 0.24169426078821, \hat{\beta} = \frac{\hat{\alpha}}{4}, \hat{\gamma} = 0.12915286960590$. Hence we have the following temporal operation:

\begin{align*}
m = 1 & \quad U_*^{(1)} = U^n, \\
& \quad U^{(1)} = U_1^{(1)} + \tau \hat{a}_{11} A^{-1} BU^{(1)}, \tag{3.8}
\end{align*}

\begin{align*}
m = 2 & \quad U_*^{(2)} = U^n + \tau \hat{a}_{21} A^{-1} L(U^{(1)}), \\
& \quad U^{(2)} = U_2^{(2)} + \tau \hat{a}_{21} A^{-1} BU^{(1)} + \tau \hat{a}_{22} A^{-1} BU^{(2)}, \tag{3.9}
\end{align*}

\begin{align*}
m = 3 & \quad U_*^{(3)} = U^n + \tau \hat{a}_{31} A^{-1} L(U^{(1)}) + \tau \hat{a}_{32} A^{-1} L(U^{(2)}), \\
& \quad U^{(3)} = U_3^{(3)} + \tau \hat{a}_{31} A^{-1} BU^{(1)} + \tau \hat{a}_{32} A^{-1} BU^{(2)} + \tau \hat{a}_{33} A^{-1} BU^{(3)},
\end{align*}

\begin{align*}
m = 4 & \quad U_*^{(4)} = U^n + \tau \hat{a}_{41} A^{-1} L(U^{(1)}) + \tau \hat{a}_{42} A^{-1} L(U^{(2)}) + \tau \hat{a}_{43} A^{-1} L(U^{(3)}), \\
& \quad U^{(4)} = U_4^{(4)} + \tau \hat{a}_{41} A^{-1} BU^{(1)} + \tau \hat{a}_{42} A^{-1} BU^{(2)} + \tau \hat{a}_{43} A^{-1} BU^{(3)} + \tau \hat{a}_{44} A^{-1} BU^{(4)}.
\end{align*}
4 Numerical results

In this section, we will discuss computational results of the scheme (3.9) on some numerical examples for the solitary wave solution and shock wave solution of Rosenau-KdV equation and Rosenau-KdV-RLW equation.

Example 1. Consider Rosenau-KdV equation (1.2) with parameters $\delta = 0, \nu = 1, \alpha = 1, \theta = 1, \varepsilon = \frac{1}{2}, p = 2$:

$$u_t + u_{xxxx} + u_{xxx} + u_x + \left(\frac{1}{2} u^2\right)_x = 0, \quad x \in [-70, 100], \quad t \in [0, T].$$

and choose the initial condition to be $u(x, 0) = M \text{sech} \left(\frac{4}{p-1}(W x)\right)$, so that the analytical solitary wave solution of Rosenau-KdV equation is $u(x, t) = M \text{sech} \left(\frac{4}{p-1}[W(x-Vt)]\right)$ as in [2] with wave width $W = \frac{p-1}{p+1} \left[-\alpha \nu (p^2+2p+5)+\sqrt{\alpha^2 \nu^2 (p^2+2p+5)^2+16 \theta^2 \nu (p+1)^2}\right]^\frac{1}{2}$, wave velocity $V = \frac{\theta (p-1)^2}{4 \nu W^2 (p^2+2p+5)}$, and wave amplitude $M = \left[\frac{-\alpha \nu (p+3)(3p+1)}{16 \nu (p+1)(p^2+2p+5)}\right]^\frac{1}{p-1}$.

Errors and rates of convergence in terms of $L_1$ and $L_\infty$ at $T = 20$ for $\tau = CFL \cdot h$ with $CFL = 1$ in interval $x \in [-70, 100]$ are listed in Table 4.1 for Example 1. The third order accuracy of the numerical method is achieved as we expected in the theoretical procedure, and works well with large time step. We can observe from the left of Figure 4.1 that the solitary wave curve matches excellently with exact solution when $h = \tau = 0.1$ at $T = 20$. From the right of Figure 4.1, it can be seen that error mostly generates at two sides of the solitary wave.

We compare the $L_\infty$ errors of our scheme with the results of other three numerical schemes [11, 12] under various mesh steps $h = \tau$ at $T = 20$ in Table 4.2. The better computational accuracy of the present scheme can be seen with the smallest error among other schemes referred above. The solitary wave graphs at $T = 10, 20$ agree with the one at $T = 0$ quite well. The solitary wave curve propagates with constant speed $V$ to the right through time $T$ in Figure 4.2.

To observe the effect of power law nonlinear term to the solitary wave of Rosenau-KdV equation, we draw the wave curves for $p = 2, 4, 6, 8, 10$ and $\varepsilon = \frac{1}{p}$ with $h = \tau = 0.1$ at $T = 20$. 
Figure 4.1: Wave graph of $u(x, t)$ at $T = 20$ and numerical solution of Rosenau-KdV equation with $h = \tau = 0.1$ at $T = 20$ (left) and error (right) for Example 1.

Figure 4.2: Numerical solution of Rosenau-KdV equation with $h = \tau = 0.1$ at $T = 0, 10, 20$ for Example 1.

Figure 4.3: Numerical solution of Rosenau-KdV equation with $p = 2, 4, 6, 8, 10, \epsilon = \frac{1}{p}$ and $h = \tau = 0.1$ at $T = 20$ for Example 1.
Table 4.1: Errors and rates of convergence when $CFL = 1, h = \tau$ at $T = 20$ for Example 1.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L_1$ order</th>
<th>$L_\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.2085e-05</td>
<td>4.7295e-04</td>
</tr>
<tr>
<td>0.1</td>
<td>3.7105e-06</td>
<td>5.4363e-05</td>
</tr>
<tr>
<td>0.05</td>
<td>4.1275e-07</td>
<td>5.9524e-06</td>
</tr>
<tr>
<td>0.025</td>
<td>4.2519e-08</td>
<td>6.0110e-07</td>
</tr>
</tbody>
</table>

Table 4.2: Comparison of $L_\infty$ errors at $T = 20$ for Example 1.

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<thead>
<tr>
<th>Scheme</th>
<th>$h = \tau = 0.2$</th>
<th>$h = \tau = 0.1$</th>
<th>$h = \tau = 0.05$</th>
<th>$h = \tau = 0.025$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme <a href="-%5Cfrac%7B1%7D%7B4%7D">11</a></td>
<td>1.0192e-03</td>
<td>2.5411e-04</td>
<td>6.3501e-05</td>
<td>1.5876e-05</td>
</tr>
<tr>
<td>Scheme <a href="%5Cfrac%7B1%7D%7B3%7D">11</a></td>
<td>4.9510e-04</td>
<td>1.2372e-04</td>
<td>3.0934e-05</td>
<td>7.7336e-05</td>
</tr>
<tr>
<td>Scheme(3.9)</td>
<td>4.7295e-04</td>
<td>5.4363e-05</td>
<td>5.9524e-06</td>
<td>6.0110e-07</td>
</tr>
</tbody>
</table>

Table 4.3: $L_1, L_\infty$ errors of numerical solutions for Rosenau-KdV equation with $h = \tau, T = 20, \varepsilon = \frac{1}{p}$ for example 1.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$h$</th>
<th>$L_1$</th>
<th>$L_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3.2085e-05</td>
<td>3.7105e-06</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1.2393e-04</td>
<td>1.5684e-05</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>1.5405e-04</td>
<td>1.9329e-05</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>1.5433e-04</td>
<td>1.9268e-05</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1.4411e-04</td>
<td>1.8034e-05</td>
</tr>
</tbody>
</table>

in Figure 4.3, The wave amplitude and width are increasing while $p$ increases. We compute $L_1, L_\infty$ errors for $p = 2, 4, 6, 8, 10$ and $\varepsilon = \frac{1}{p}$ at $T = 20$ on three different meshes in Table 4.3 and also achieve third-order convergence in each case.

Next, We refer shock wave solutions of Rosenau-KdV equation from [3] which is available only for two particular values of power law nonlinearity parameter $p = 3, 5$. Our scheme simulates this wave phenomena efficiently with its essentially non-oscillatory property.
Figure 4.4: Wave graph of $u(x, t)$ at $T = 10$ and numerical solution of Rosenau-KdV equation with $h = \tau = 0.1, p = 5$ at $T = 10$ (left) and error (right) for Example 2.

**Example 2.** Consider Rosenau-KdV equation (1.2) with parameters $\delta = 0, \nu = -10, \alpha = 0.05, \theta = 0.001, \epsilon = -5, p = 5$:

$$u_t + 0.05u_x + 0.001u_{xxx} - 10u_{xxxx} - 5(u^5)_x = 0, \quad x \in [-10, 10], t \in [0, T].$$

and choose the initial condition to be $u_0(x) = M \tanh(Wx)$, so that the analytical shock wave solution of Rosenau-KdV equation for $p = 5$ is $u(x, t) = M \tanh[W(x - Vt)]$ as in [3] with $W = \frac{1}{2} \left[ \frac{5\alpha}{3\theta} - \frac{1}{3\theta} \sqrt{\frac{\alpha^2 + 25\alpha^2 \nu}{\nu}} \right]^{\frac{1}{2}}, \quad V = \frac{\alpha - 26W^2}{16W^2 + 1}, \quad$ and $M = \frac{W}{\epsilon^{\frac{5}{4}}}.$

These parameters have to be chosen carefully to make sure that the three quantities are all real. In Table 4.4, we show errors and rates of convergence to highlight the efficiency of the WENO reconstruction for shock wave in the case of $p = 5$. Figure 4.4 displays the shock wave at $T = 10$ with $h = \tau = 0.1$ on the left and error on the right. As we can see there is no oscillatory nearby the stiff region.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L_1$ order</th>
<th>$L_\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.5912e-03</td>
<td>7.8227e-03</td>
</tr>
<tr>
<td>0.1</td>
<td>7.2593e-04</td>
<td>1.8357</td>
</tr>
<tr>
<td>0.05</td>
<td>1.2977e-04</td>
<td>2.4839</td>
</tr>
<tr>
<td>0.025</td>
<td>1.6055e-05</td>
<td>3.0148</td>
</tr>
</tbody>
</table>
Figure 4.5: Wave graph of \( u(x, t) \) at \( T = 10 \) and numerical solution of Rosenau-KdV equation with \( h = \tau = 0.1, p = 3 \) at \( T = 10 \) (left) and error (right) for Example 3.

**Example 3.** Consider Rosenau-KdV equation (1.2) with parameters \( \delta = 0, \nu = -10, \alpha = 0.4, \theta = 0.01, \epsilon = -3, p = 3 \):

\[
    u_t + 0.4u_x + 0.01u_{xxx} - 10u_{xxxx} - 3(u^3)_x = 0, \quad x \in [-10, 10], t \in [0, T].
\]

and choose the initial condition to be \( u_0(x) = M \tanh^2(Wx) \), so that the analytical shock wave solution of Rosenau-KdV equation for \( p = 3 \) is \( u(x, t) = M \tanh^2[W(x - Vt)] \) as in [3] with

\[
    W = \frac{1}{2} \left[ \frac{10\alpha}{23\theta} - \frac{1}{23\theta} \sqrt{\frac{100\alpha^2\nu + 46\theta^2}{\nu}} \right]^{\frac{1}{2}}, \quad V = \frac{\alpha - 8\theta W^2}{136\nu W^4 + 1}, \quad M = 2W^2 \left( \frac{30V\nu}{\epsilon} \right)^{\frac{1}{2}}.
\]

In Table 4.5, we give error and rate of convergence for shock wave when \( p = 3 \). Obviously here we achieve order that smaller than three at first, but it will converge to three eventually as the mesh is refined. Figure 4.5 displays the shock wave at \( T = 10 \) when \( h = \tau = 0.1 \) on the left and error on the right.

Table 4.5: Errors and rates of convergence when \( CFL = 1, h = \tau \) at \( T = 10 \) for Example 3.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( L_1 ) order</th>
<th>( L_\infty ) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.1529e-03</td>
<td>5.7931e-03</td>
</tr>
<tr>
<td>0.1</td>
<td>7.5346e-04</td>
<td>1.5147</td>
</tr>
<tr>
<td>0.05</td>
<td>1.3853e-04</td>
<td>2.4433</td>
</tr>
<tr>
<td>0.025</td>
<td>1.7016e-05</td>
<td>3.0252</td>
</tr>
</tbody>
</table>
Example 4. Consider Rosenau-KdV-RLW equation (1.3) with parameters $\delta = -1, \nu = 1, \alpha = 1, \theta = 1, \varepsilon = \frac{1}{2}, p = 2$:

$$u_t - u_{xxx} + u_{xxxxx} + u_x + u_{xxx} + \frac{1}{2}(u^2)_x = 0, \quad x \in [-40, 60], t \in [0, T].$$

and choose the initial condition to be $u(x, 0) = Msech_{\frac{4}{1-p}}(Wx)$, so that the analytical solitary wave solution of Rosenau-KdV-RLW equation is $u(x, t) = Msech_{\frac{4}{1-p}}[W(x-Vt)]$ as in [9] with

$$D = \sqrt{\alpha^2 \nu^2 (p^2 + 2p + 5)^2 + 16(p + 1)^2 \theta \nu(\theta - \alpha \delta)}, \text{ wave width } W = \frac{4}{p+1} \sqrt{\frac{D-(p^2+2p+5)\theta \nu}{320 \nu}},$$

wave speed $V = \frac{\theta (p-1)^2}{(p-1)^2 \delta + 4 \nu W^2 (p^2 + 2p + 5)}$, and amplitude $M = \left[\frac{8(p+1)(p+3)(3p+1)\theta \nu W^4}{\varepsilon (p-1)^2 (p-1)^2 \delta + 4 (p^2 + 2p + 5) \nu W^2}\right]^{\frac{1}{p+1}}$.

The $L_\infty$ errors of the numerical solutions at $T = 10$ under various mesh steps $h = \tau$ are listed in Table 4.6 and compare with other three types of schemes studied earlier about the same equation, which shows that our scheme has the smallest error in any cases.

On the left of Figure 4.6, the numerical wave curve totally matches with the analytical solitary solution at $T = 10$ with mesh $h = \tau = 0.1$ over the interval $x \in [-40, 60]$ and the corresponding distribution of the error is drawn for solitary wave in the right of Figure 4.6.

As shown in Table 4.7, the third-order convergence of the numerical solutions is verified at $T = 10$ for the solitary wave problem of the Rosenau-KdV-RLW equation. In Figure 4.7, perspective views of the traveling solutions are graphed at various time levels for $h = \tau = 0.1$.

In order to observe the effect of power law nonlinear term to the solitary wave of Rosenau-KdV-RLW equation, The $L_1, L_\infty$ errors and third-order convergence for $p = 2, 4, 6, 8, 10$ and $\varepsilon = \frac{1}{p}$ on three different mesh are listed in Table 4.8. We draw the wave curves for these $p$ at $T = 10$ with $CFL = 1, h = \tau = 0.1$ and $\varepsilon = \frac{1}{p}$ in the interval $x \in [-40, 60]$ for give further description in Figure 4.8. It can be observe that wave amplitude and speed decreases along with $p$ increases, this also fits the power law.

Based on earlier studies on the shock solution of the Rosenau-KdV equation, the shock wave solutions for the Rosenau-KdV-RLW equation were extracted by balancing principle only for $p = 3$ and $p = 5$ in [9]. Here we will review related formulation for wave amplitude, width, velocity mentioned in [9, 10], and then simulate both cases numerically as example.
Figure 4.6: Wave graph of $u(x, t)$ at $T = 10$ and numerical solution of Rosenau-KdV-RLW equation with $h = \tau = 0.1$ at $T = 10$ (left) and error (right) for Example 4.

Figure 4.7: Numerical solution of Rosenau-KdV-RLW equation with $h = \tau = 0.1$ at $T = 2, 4, 6, 8, 10$ for Example 4.

Figure 4.8: Numerical solution of Rosenau-KdV-RLW equation with $p = 2, 4, 6, 8, 10$, $\epsilon = \frac{1}{p}$ and $h = \tau = 0.1$ at $T = 10$ for Example 4.
Table 4.6: The Comparison of $L_\infty$ errors with $CFL = 1, h = \tau$ at $T = 10$ between four different schemes for Example 4.

<table>
<thead>
<tr>
<th>$(h, \tau)$</th>
<th>(0.4,0.4)</th>
<th>(0.2,0.2)</th>
<th>(0.1,0.1)</th>
<th>(0.05,0.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme [11] $\left(\frac{1}{3}\right)$</td>
<td>6.39957e-2</td>
<td>1.52505e-2</td>
<td>3.79081e-3</td>
<td>9.48668e-4</td>
</tr>
<tr>
<td>Scheme [11] $\left(\frac{2}{3}\right)$</td>
<td>1.20316e-1</td>
<td>3.03968e-2</td>
<td>7.61986e-3</td>
<td>1.90703e-3</td>
</tr>
<tr>
<td>Scheme [12]</td>
<td>1.11946e-1</td>
<td>2.88404e-2</td>
<td>7.2695e-3</td>
<td>1.81991e-3</td>
</tr>
<tr>
<td>Scheme(3.9)</td>
<td>5.4870e-02</td>
<td>7.6075e-03</td>
<td>9.8854e-04</td>
<td>1.2479e-04</td>
</tr>
</tbody>
</table>

Table 4.7: Errors and rates of convergence with $CFL = 1, h = \tau$ at $T = 10$ for Example 4.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L_1$</th>
<th>rate</th>
<th>$L_\infty$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>8.4789e-04</td>
<td>7.6075e-03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.1027e-04</td>
<td>2.9428</td>
<td>9.8854e-04</td>
<td>2.9441</td>
</tr>
<tr>
<td>0.05</td>
<td>1.3908e-05</td>
<td>2.9871</td>
<td>1.2479e-04</td>
<td>2.9858</td>
</tr>
<tr>
<td>0.025</td>
<td>1.7632e-06</td>
<td>2.9912</td>
<td>1.5496e-05</td>
<td>3.0096</td>
</tr>
</tbody>
</table>

Table 4.8: $L_1, L_\infty$ errors of numerical solutions for Rosenau-KdV-RLW equation with $h = \tau, T = 10, \varepsilon = \frac{1}{p}$ for Example 4.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L_1$</th>
<th>$L_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>8.4789e-04</td>
<td>7.6075e-03</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1027e-04</td>
<td>2.9428</td>
</tr>
<tr>
<td>0.05</td>
<td>1.3908e-05</td>
<td>2.9871</td>
</tr>
<tr>
<td>0.025</td>
<td>1.7632e-06</td>
<td>2.9912</td>
</tr>
</tbody>
</table>

Example 5. Consider Rosenau-KdV-RLW equation (1.3) with parameters $\delta = 1, \nu = -0.001, \alpha = 0.01, \theta = 0.001, \varepsilon = -1, p = 3$:

$$u_t + 0.01u_x + 0.001u_{xxx} + u_{xxt} - 0.001u_{xxxx} - (u^3)_x = 0, \quad x \in [-10, 10], t \in [0, T].$$

and choose the initial condition to be $u_0(x) = M \tanh^2(Wx)$, so that the analytical shock wave solution of Rosenau-KdV-RLW equation for $p = 3$ is $u(x,t) = M \tanh^2[W(x-Vt)]$ as
Figure 4.9: Wave graph of $u(x,t)$ at $T = 10$ and numerical solution of Rosenau-KdV-RLW equation with $h = \tau = 0.1, p = 3$ at $T = 10$ (left) and error (right) for Example 5.

In [9] with $W = \left[ \frac{100\alpha^2\nu^2 + 466\nu(\theta - \alpha\delta)}{929\nu} \right]^\frac{1}{2}$, $V = \frac{\alpha - 86W^2}{136W^4 - 88W^2 + 1}$, and $M = 2W^2(\frac{30VW}{\varepsilon})^\frac{1}{2}$.

In Table 4.9, the error comparisons in $L^\infty, L_1$ are obtained by present method for shock wave solution in the case of $p = 3$ of the Rosenau-KdV-RLW equation in interval $x \in [-10, 10]$ with $h = \tau = 0.2, 0.1, 0.05, 0.025$ respectively and the simulations are run up to time $T = 10$ to obtain the error norms. It can be easily found that the errors are small, and the third-order convergence of the numerical solutions are also verified. From Figure 4.9, we can catch the point that numerical solution fits with exact one, and numerical method approximate the exact solution even in stiff concave region successfully.

Table 4.9: Errors and rates of convergence with $CFL = 1, h = \tau$ at $T = 10$ for Example 5.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L_1$</th>
<th>rate</th>
<th>$L^\infty$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>8.0600e-06</td>
<td></td>
<td>1.6909e-04</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>7.8356e-07</td>
<td>3.3627</td>
<td>2.8062e-05</td>
<td>2.5911</td>
</tr>
<tr>
<td>0.05</td>
<td>6.7878e-08</td>
<td>3.5290</td>
<td>2.5392e-06</td>
<td>3.4662</td>
</tr>
<tr>
<td>0.025</td>
<td>6.9675e-09</td>
<td>3.2842</td>
<td>2.3362e-07</td>
<td>3.4421</td>
</tr>
</tbody>
</table>

Example 6. Consider Rosenau-KdV-RLW equation (1.3) with parameters $\delta = 1, \nu = -10, \alpha = 0.05, \theta = 0.001, \varepsilon = -5, p = 5$:

$$u_t + u_{xxt} - 10u_{xxxxx} + 0.05u_x + 0.001u_{xxx} - 5(u^5)_x = 0 \quad x \in [-10, 10], t \in [0, T].$$
and choose the initial condition to be \( u_0(x) = M \tanh(Wx) \) so that the analytical shock wave solution of Rosenau-KdV-RLW equation for \( p = 5 \) is \( u(x, t) = M \tanh[W(x - Vt)] \) as in [9] with \( W = \left[ \frac{5\alpha\nu - \sqrt{25\alpha^2\nu^2 + 6\theta\nu(\theta - \alpha)}}{12\theta\nu} \right]^\frac{1}{2}, \quad V = \frac{\alpha - 2\theta W^2}{16\nu W^4 - 29W^2 + 1}, \) and \( M = W \left( \frac{24V\nu}{\varepsilon} \right)^\frac{1}{4}. \)

The computation of error and order is completed at time \( t = 10 \) when \( CFL = 1, h = \tau \) on various mesh in interval \( x \in [-10, 10] \) and displayed in Table 4.10. The numerical shock wave curve of Rosenau-KdV-RLW equation for \( p = 5 \) is agree with exact solution with no oscillatory near the point \( x = 0 \) when \( h = \tau = 0.1 \) at \( T = 10 \) on the left of Figure 4.10.

Table 4.10: Errors and rates of convergence with \( CFL = 1, h = \tau \) at \( T = 10 \) for Example 6.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( L_1 )</th>
<th>rate</th>
<th>( L_\infty )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.1778e-03</td>
<td></td>
<td>5.2701e-03</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>6.6553e-04</td>
<td>1.7103</td>
<td>1.8419e-03</td>
<td>1.5167</td>
</tr>
<tr>
<td>0.05</td>
<td>1.2634e-04</td>
<td>2.3972</td>
<td>3.6413e-04</td>
<td>2.3387</td>
</tr>
<tr>
<td>0.025</td>
<td>1.5887e-05</td>
<td>2.9914</td>
<td>4.6009e-05</td>
<td>2.9845</td>
</tr>
</tbody>
</table>

5 Concluding Remark

To solve the solitary wave and shock wave problem of Rosenau-KdV equation and Rosenau-KdV-RLW equation, we use the third-order finite difference WENO reconstruc-
tion for advection terms, and central finite difference method for other terms in spatial
discretization, then we use third-order SSP IMEX Runge-Kutta method for time discretiza-
tion, in which the advection terms are treated by explicitly and remaining terms are treated
by implicitly. In order to verify the effectiveness of the numerical scheme, some numerical
examples are given for numerical experiment. Numerical simulations show that the method
is very efficient with the advantages of non-oscillatory and loosely-restricted CFL condition.

References

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