

FAST PRECONDITIONERS FOR TOTAL VARIATION DEBLURRING WITH ANTI-REFLECTIVE BOUNDARY CONDITIONS

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Abstract. In recent works several authors have proposed the use of precise boundary conditions (BCs) for blurring models and they proved that the resulting choice (Neumann or reflective, anti-reflective) leads to fast algorithms both for deblurring and for detecting the regularization parameters in presence of noise. When considering a symmetric point spread function, the crucial fact is that such BCs are related to fast trigonometric transforms.

In this paper we combine the use of precise BCs with the Total Variation (TV) approach in order to preserve the jumps of the given signal (edges of the given image) as much as possible. We consider a classic fixed point method with a preconditioned Krylov method (usually the conjugate gradient method) for the inner iteration. Based on fast trigonometric transforms, we propose some preconditioning strategies which are suitable for reflective and anti-reflective BCs. A theoretical analysis motivates the choice of our preconditioners and an extensive numerical experimentation is reported and critically discussed. Numerical tests show that the TV regularization with anti-reflective BCs implies not only a reduced analytical error, but also a lower computational cost of the whole restoration procedure over the other BCs.

Keywords: Sine algebra of type I (τ algebra), reflective and anti-reflective BCs, total variation, preconditioning.

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1. Introduction. We are concerned with specific linear algebra/matrix theory aspects of the vast field of inverse problems [17, 18] which model the blurring of signals and images ($2D$ or dD with $d \geq 3$). Here the goal is to reconstruct the real object from its blurred and noisy version and this goal is a classical one in astronomical imaging, medical imaging, geosciences, etc. [5].

The blurring model is assumed to be space-invariant, i.e., the point spread function (PSF) is represented by a specific bivariate function $h(x - y)$ for some univariate function $h(\cdot)$ [19]. According to the linear models described in the literature [17], the observed signal or image v and the original signal or image u are described by

$$(1.1) \quad v(x) = \mathcal{H}u(x) + \eta(x) := \int_{\Omega} h(x - s)u(s)ds + \eta(x), \quad x \in \Omega,$$

where the kernel h is the PSF and η denotes the noise. The problem (1.1) is ill-posed since the operator \mathcal{H} is compact [17]. Therefore, the approximation/discretization matrix of \mathcal{H} is usually increasingly ill-conditioned when the number n of pixels becomes large. In addition, the size of the subspace associated with small eigenvalues, which substantially intersects the high frequencies, is large and proportional to the size of the matrix. Thus, we cannot directly solve $\mathcal{H}u = d$, since the small perturbations, represented by the noise η with important high frequency components due to its probabilistic nature, would be amplified unacceptably.

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To remedy the essential ill-conditioning of problem (1.1), one may employ regularization methods. The Total Variation (TV) regularization approach is a good choice for restoring edges of the original signals [25]. Rudin, Osher, and Fatemi [25] gave the total variation functional in the form

$$(1.2) \quad J_{TV}(u) := \int_{\Omega} |\nabla u| dx,$$

where $|\cdot|$ denotes the $L1$ -norm. We note that the $L1$ -norm $|\cdot|$ is not differentiable at zero. To avoid the non-differentiability, Acar and Vogel [2] considered the following minimization problem

$$(1.3) \quad \min_u \|\mathcal{H}u - v\|_{L^2(\Omega)} + \alpha \int_{\Omega} \sqrt{|\nabla u|^2 + \beta^2} dx,$$

where α, β are positive parameters. Notice that the penalty term $\int_{\Omega} \sqrt{|\nabla u|^2 + \beta^2} dx$ converges to $J_{TV}(u)$ as $\beta \rightarrow 0$. In other words, the penalty term is a differentiable regularized version of $J_{TV}(u)$. The corresponding Euler-Lagrange equation for (1.3) is given by

$$(1.4) \quad g(u) := \mathcal{H}^*(\mathcal{H}u - v) - \alpha \mathcal{L}_u(v) = 0$$

under the zero Neumann boundary conditions, where \mathcal{H}^* is the adjoint of \mathcal{H} and $\mathcal{L}_u(y) := -\nabla \cdot (1/\sqrt{|\nabla u|^2 + \beta^2} \nabla y)$ is the differential operator appearing in (1.4) and comes from the regularized penalty term. The lagged diffusivity fixed point (FP) iteration is given by

$$(1.5) \quad A_{u^k} u^{k+1} \equiv (H^* H + \alpha L(u^k)) u^{k+1} = H^* v, \quad k = 0, 1, \dots,$$

where H and $L(u^k)$ denote the discretization matrix of \mathcal{H} and \mathcal{L}_{u^k} , respectively, which was proposed by Vogel and Oman [33] for solving (1.4). One may solve (1.5) by the preconditioned conjugate gradient (PCG) method [16, Alg. 10.3.1].

In [8] the authors proposed a cosine preconditioner when H is a Toeplitz matrix, i.e., in the case of zero-Dirichlet boundary conditions (BCs). However, the choice of such BCs induces remarkable pathologies in the quality of the restored images, which should be avoided or at least minimized. In reality, using classical BCs such as periodic or zero-Dirichlet may imply, when the background is not uniformly black, disturbing Gibbs phenomena called ringing effects [19, 22, 27].

The novelty of this paper is represented by the choice of appropriate BCs, in order to reduce the ringing effects, and in the related matrix/numerical analysis. The latter will affect the algebraic expression of H , while for $L(u^k)$ the choice of BCs in the Euler-Lagrange equation for (1.4) seems to impose Neumann BCs. In this context, the idea is to combine the application of anti-reflective BCs, already studied for their precision with plain regularization methods like Tikhonov and Landweber [4, 11, 12, 24, 27, 29], with the more sophisticate TV regularization. In other words, the first aim consists of checking how to reduce the ringing effects and the over-smoothing of the edges simultaneously. Next, we want to study the use of preconditioners based on innovative fast transforms in the setting of Krylov methods when the real problem is modeled by a symmetric PSF. The final goal is to combine the precision of the reconstruction with highly efficient numerical procedures. We study some preconditioning techniques and give the theoretical explanations of different proposals. An effective preconditioner for reflective BCs is inspired by the work in [8], while, for the anti-reflective BCs,

Thus the blurred signal v is determined not only by u , but also by $(u_{-m+1}, \dots, u_0)^T$ and $(u_{n+1}, \dots, u_{n+m})^T$ and the linear system (2.1) is underdetermined. To overcome this, we make certain assumptions (called boundary conditions) on the unknown boundary data u_{-m+1}, \dots, u_0 and u_{n+1}, \dots, u_{n+m} in such a way that the number of unknowns equals the number of equations.

For the *zero-Dirichlet* BCs, we assume that the data outside u are zero, i.e., we set $u_{1-j} = u_{n+j} = 0$ for $j = 1, \dots, m$. Then, (2.1) becomes $Au = v$, where A is Toeplitz. For the *periodic* BCs, we assume that the signal u is extended by periodicity. More precisely, we set $u_{1-j} = u_{n-j+1}$ and $u_{n+j} = u_j$ for $j = 1, \dots, m$. It follows that (2.1) becomes $Au = v$, where A is circulant and hence it can be diagonalized by the Discrete Fourier Transform (DFT) (see [19]).

For the *Neumann or reflective* BCs, we assume that the data outside u are a reflection of the data inside u . Discretization of the Neumann BCs can be centered at a meshpoint or a midpoint, different choices do not affect the approximation error but only the structure of the matrix. More in detail, the eigenvectors are related to different discrete cosine transforms [30]. In this paper we use only the midpoint discretization that leads to the standard discrete cosine transform of type II (DCT-2) previously considered in [22]. Therefore, we set $u_{1-j} = u_j$ and $u_{n+j} = u_{n+1-j}$ for all $j = 1, \dots, m$ in (2.1). Thus (2.1) becomes $Au = v$, where A is neither Toeplitz nor circulant but a special n -by- n Toeplitz plus Hankel matrix which is diagonalized by the discrete cosine transform provided that the blurring function h is symmetric, i.e., $h_j = h_{-j}$ for all j . It follows that the above system can be solved by using three fast cosine transforms (FCTs) in $O(n \log n)$ operations [22]. This approach is computationally interesting since the FCT requires only real operations and is about twice as fast as the fast Fourier transform (FFT). We note that the reflection ensures the continuity of the signal and the error is linear in the discretization step (the linear error can be easily seen by applying a Taylor expansion [22]). Therefore, we usually observe a reduction of the boundary artifacts with respect to zero-Dirichlet and periodic BCs.

For the *anti-reflective* BCs, we assume that the data outside u are an anti-reflection of the data inside u . According to the terminology employed in [30], we perform a midpoint anti-reflection like for the Neumann BCs. More precisely, if x is a point outside the domain and x^* is the closest boundary point, then we have $x = x^* - \delta x$ and the quantity $u(x)$ is approximated by $u(x^*) - (u(x^* + \delta x) - u(x^*))$. Consequently, we set

$$(2.2) \quad \begin{aligned} u_{1-j} &= u_1 - (u_{j+1} - u_1) = 2u_1 - u_{j+1}, & \text{for all } j = 1, \dots, m, \\ u_{n+j} &= u_n - (u_{n-j} - u_n) = 2u_n - u_{n-j}, & \text{for all } j = 1, \dots, m \end{aligned}$$

in (2.1). By a Taylor expansion, the anti-reflection (2.2) ensures a C^1 continuity of the signal and the error is quadratic in the discretization step [27]. Usually, the boundary artifacts are also reduced with respect to reflective BCs.

Imposing the anti-reflection (2.2), (2.1) becomes $Au = v$, where A is a Toeplitz plus Hankel plus a rank-2 correction matrix, where the correction is placed at the first and the last column. Furthermore, in [4] the authors proved that if h is symmetric then $A = T_n \Lambda T_n^{-1}$, with

$$(2.3) \quad T_n = \begin{bmatrix} 1 & 0 & 0 \\ p & S_{n-2} & Jp \\ 0 & 0 & 1 \end{bmatrix}, \quad T_n^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -S_{n-2}p & S_{n-2} & -S_{n-2}Jp \\ 0 & 0 & 1 \end{bmatrix},$$

where J is the flip matrix, S_{n-2} is the sine transform matrix of order $n-2$, and $p_j = 1 - j/(n-1)$ so that the first column vector is exactly the sampling of the function $1-x$ on the grid $j/(n-1)$ for $j = 0, \dots, n-1$. Finally, Λ is a diagonal matrix given by suitable samplings of the function

$$(2.4) \quad \hat{h}(y) = \sum h_j \exp(ijy),$$

which is the symbol of the PSF. That is,

$$(2.5) \quad \Lambda = \text{diag}_{j=1, \dots, n} \left(\hat{h}(y_j) \right), \quad S_m = \sqrt{\frac{2}{m+1}} \left(\sin \left(\frac{ji\pi}{m+1} \right) \right)_{i,j=1}^m,$$

where

$$(2.6) \quad y_j = \frac{(j-1)\pi}{n-1}, \text{ for } j = 1, \dots, n-1, \quad \text{and} \quad y_n = 0.$$

As a consequence, a generic system $Au = v$ can be solved within $O(n \log n)$ real operations by resorting to the application of three fast sine transforms (FSTs) (refer to [27]), where each FST is computationally as cheap as a generic FCT. There is a suggestive functional interpretation of the transform T_n : it can be functionally interpreted as a linear combination of sine functions and of linear polynomials (whose use is exactly required for imposing the C^1 continuity at the borders). This intuition becomes evident in the expression of T_n . Indeed, the j -th row of T_n is given by

$$(2.7) \quad \left(1 - \frac{y_j}{\pi}, \sin(y_j), \dots, \sin((n-2)y_j), \frac{y_j}{\pi} \right) \cdot \text{diag} \left(1, \sqrt{\frac{2}{n-1}} I_{n-2}, 1 \right),$$

where y_j is defined in (2.6) for $j = 1, \dots, n$ and it is a suitable gridding of $[0, \pi]$.

Now, we introduce the two-dimensional case. For the Neumann or reflective BCs, the blurring matrix is a block Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks and can be diagonalized by the two-dimensional FCTs, which are tensor products of one-dimensional FCTs, in $O(n^2 \log n)$ operations provided that h is quadrantly symmetric, i.e., $h_{i,j} = h_{-i,j} = h_{i,-j} = h_{-i,-j}$ (refer to [22]).

For the anti-reflective BCs a point outside the domain is anti-reflected to the closest boundary point first in one direction and then in the other direction. In particular, we set

$$\begin{aligned} u_{1-j,\phi} &= 2u_{1,\phi} - u_{j+1,\phi}, & u_{n+j,\phi} &= 2u_{n,\phi} - u_{n-j,\phi}, & \text{for } 1 \leq j \leq m, 1 \leq \phi \leq n, \\ u_{\psi,1-j} &= 2u_{\psi,1} - u_{\psi,j+1}, & u_{\psi,n+j} &= 2u_{\psi,n} - u_{\psi,n-j}, & \text{for } 1 \leq j \leq m, 1 \leq \psi \leq n. \end{aligned}$$

When both indices lie outside the range $\{1, \dots, n\}$ (this happens close to the 4 corners of the given image), we set

$$\begin{aligned} u_{1-i,1-j} &= 4u_{1,1} - 2u_{1,j+1} - 2u_{i+1,1} + u_{i+1,j+1}, \\ u_{1-i,n+j} &= 4u_{1,n} - 2u_{1,n-j} - 2u_{i+1,n} + u_{i+1,n-j}, \\ u_{n+i,1-j} &= 4u_{n,1} - 2u_{n,j+1} - 2u_{n-i,1} + u_{n-i,j+1}, \\ u_{n+i,n+j} &= 4u_{n,n} - 2u_{n,n-j} - 2u_{n-i,n} + u_{n-i,n-j}, \end{aligned}$$

for $1 \leq i, j \leq m$. If the blurring function (PSF) h is quadrantly symmetric, then the blurring matrix is a block Toeplitz-plus-Hankel-plus-2-rank-correction matrix with Toeplitz-plus-Hankel-plus-2-rank-correction blocks and can be diagonalized

by the two-dimensional anti-reflective transforms, which are tensor products of one-dimensional anti-reflective transforms T_n , in $O(n^2 \log n)$ real operations; see for instance [4]. In the following we will assume a symmetric (quadrantally symmetric in 2D) PSF since reflective and anti-reflective BCs can be diagonalized by fast transforms only in such case. However, in the nonsymmetric case, even if the blurring matrix can not be diagonalized by fast transforms, the matrix-vector product can be done again in $O(n^2 \log n)$ by FFTs. Moreover, many practical blurs have symmetry like the celebrated Gaussian blur, widely used in several contexts.

3. Optimal preconditioners with different boundary conditions. The optimal preconditioner for a matrix A aims to find an approximation which minimizes $\|B - A\|_F$ over all B in a set of matrices for the Frobenius matrix norm $\|\cdot\|_F$: the typical set of matrices is formed by considering an algebra of matrices which are simultaneously diagonalized by a given unitary transform. The main novelty in our context is represented by the fact that the anti-reflective matrices with symmetric PSFs form a commutative algebra associated to a non-unitary transform. The anti-reflective algebra poses nontrivial difficulties that are treated in the sequel of the paper. The optimal circulant preconditioner was originally given in [9]. The optimal sine transform preconditioner was presented in [7]. The optimal cosine transform preconditioner was provided in [6]. In this section, we construct the optimal reflective BCs preconditioner and the optimal anti-reflective BCs preconditioner for (1.5) and the optimal reblurring preconditioner for the reblurring equation (3.8) below instead of (1.5). Some of these preconditioning techniques are inspired from the idea proposed in [6, 8] for zero-Dirichlet BCs.

3.1. One-dimensional problems. For the one-dimensional problems, we assume a symmetric and normalized PSF.

3.1.1. The reflective boundary condition. Suppose that we impose the reflective BCs on H and the zero Neumann BCs on $L(u^k)$. In this case, we propose the following reflective BCs preconditioners for (1.5). Let C_n be the n dimensional discrete cosine transform with entries

$$[C_n]_{i,j} = \sqrt{\frac{2 - \delta_{j1}}{n}} \cos\left(\frac{(2i-1)(j-1)\pi}{2n}\right), \quad i, j = 1, \dots, n,$$

where δ_{ij} is the Kronecker delta. The matrix C_n is orthogonal, i.e., $C_n C_n^T = I$. Moreover, for any n -vector w , the matrix-vector product $C_n w$ can be computed within $O(n \log n)$ real operations by the FCT.

For an n -by- n matrix A , the optimal cosine transform preconditioner is

$$c(A) = \operatorname{argmin}_{B \in \mathcal{C}} \|B - A\|_F,$$

where $\mathcal{C} = \{C_n^T \Lambda C_n : \Lambda \text{ is a real } n\text{-by-}n \text{ diagonal matrix}\}$. The operator $c(\cdot)$ is linear, preserves positive definiteness, and compresses every unitarily invariant norm. For the specific cosine algebra and a general convergence theory based on the Korovkin theorems, we refer to [8, 26].

As in [8] for Dirichlet BCs, the optimal cosine transform preconditioner (i.e., the optimal reflective BCs preconditioner) for (1.5) can be defined as

$$(3.1) \quad R = H^* H + \alpha c(L(u^k)).$$

We note that $R = c(A_{u^k})$ since $H^* = H \in \mathcal{C}$. Spectral properties of the preconditioner will be discussed in Section 4. Here, we only note that $c(L(u^k))$ is not an optimal preconditioner for $L(u^k)$ if the coefficient $(|\nabla u|^2 + \beta^2)^{-1/2}$ has large variation. In such case a diagonal scaling is necessary to obtain an effective preconditioner like $\text{diag}(L(u^k))^{1/2} c(L(u^k)) \text{diag}(L(u^k))^{1/2}$, where $\text{diag}(L(u^k))$ is the diagonal matrix whose diagonal entries are the same as that of $L(u^k)$ [28]. We note that the coefficient matrix in (1.5) is the sum of two operators. To avoid the possibly large fluctuation in the coefficient of the operator in (1.5), we define a *scaled* reflective BCs preconditioner for (1.5) by $D_R = D^{\frac{1}{2}} R D^{\frac{1}{2}}$, where R is given in (3.1) and

$$(3.2) \quad D \equiv I + \alpha \text{diag}(L(u^k)).$$

A further possibility is to employ a diagonal scaling for (1.5). As in [8], we consider the scaled equation

$$(3.3) \quad \tilde{A}_{u^k} \tilde{u}^{k+1} \equiv \left(\tilde{H}^* \tilde{H} + \alpha \tilde{L}(u^k) \right) \tilde{u}^{k+1} = \tilde{H}^* v,$$

where $\tilde{H} = H D^{-1/2}$, $\tilde{L}(u^k) = D^{-1/2} L(u^k) D^{-1/2}$, and $\tilde{u}^k = D^{1/2} u^k$. Then we propose an *unscaled* reflective BCs preconditioner for (3.3) by $R_D = \tilde{H}^* \tilde{H} + \alpha c(\tilde{L}(u^k))$, where $\tilde{H} = H c(D^{-1/2})$. If Λ_H , Λ_D and $\Lambda_{\tilde{L}}$ denote the eigenvalue matrices of H , $c(D^{-1/2})$ and $c(\tilde{L}(u^k))$, respectively, then R_D can be written as

$$R_D = C_n^T (\Lambda_H^* \Lambda_H \Lambda_D^* \Lambda_D + \alpha \Lambda_{\tilde{L}}) C_n.$$

3.1.2. The anti-reflective boundary condition. Under anti-reflective BCs for H and the zero Neumann BCs or anti-reflective BCs for $L(u^k)$, we construct the anti-reflective BCs preconditioners for (1.5) and then propose a preconditioned reblurring method. Let S_n be the n dimensional discrete sine transform of type I with entries as in (2.5). Then S_n is orthogonal and symmetric, i.e., $S_n^T = S_n$ and $S_n^2 = I$. Moreover, for any n dimensional vector w , the matrix-vector product $S_n w$ can be computed in $O(n \log n)$ real operations by the FST. Let $\sigma(z) := (z_2, \dots, z_n, 0)^T$ with $z = (z_1, \dots, z_n)^T$. Let $\mathcal{T}(z)$ be the n -by- n symmetric Toeplitz matrix whose first column is z and $\mathcal{H}(z, Jz)$ be the n -by- n Hankel matrix whose first and last column are z and Jz , respectively. It was shown that for any $B \in \tau$, there exists $z = (z_1, \dots, z_n)^T \in \mathbb{R}^n$ such that [7] $B = \mathcal{T}(z) - \mathcal{H}(\sigma^2(z), J\sigma^2(z))$. For an n -by- n matrix A , the optimal sine preconditioner is

$$(3.4) \quad s(A) = \underset{B \in \tau}{\text{argmin}} \|B - A\|_F,$$

where $\tau := \{S_n \Lambda S_n : \Lambda \text{ is a real diagonal matrix of order } n\}$. The construction of $s(A)$ requires only $O(n^2)$ operation for a general matrix A and $O(n)$ operation for a banded matrix A . Furthermore, $s(\cdot)$ is linear, preserves positive definiteness, and compresses any unitarily invariant norm (see [7, 26]).

Now, we define an optimal sine transform based preconditioner (i.e., the so-called anti-reflective BCs preconditioner) for (1.5) by

$$(3.5) \quad M = \hat{s}(H)^* \hat{s}(H) + \alpha \hat{s}(L(u^k))$$

in the sense that, for any n -by- n matrix A , $\hat{s}(A)$ is given by

$$(3.6) \quad \hat{s}(A) = \underset{B \in \hat{\tau}}{\text{argmin}} \|B - A\|_F,$$

where $\hat{\tau} = \{\hat{S}_n \Lambda \hat{S}_n : \Lambda \text{ is a real diagonal matrix of order } n \text{ and } \hat{S}_n = \text{diag}(1, S_{n-2}, 1)\}$.

PROPOSITION 3.1. *Given an n -by- n matrix A , we have*

$$\hat{s}(A) = \begin{bmatrix} A(1,1) & 0 & 0 \\ 0 & s(A(2:n-1, 2:n-1)) & 0 \\ 0 & 0 & A(n,n) \end{bmatrix}.$$

where $\hat{s}(\cdot)$ and $s(\cdot)$ are defined in (3.6) and (3.4), respectively, and $A(2:n-1, 2:n-1)$ is the submatrix of A corresponding to rows indexed from 2 to $n-1$ and columns from 2 to $n-1$.

Proof. By unitary invariance of the Frobenius norm (\hat{S}_n is unitary) we find $\|A - \hat{S}_n \Lambda \hat{S}_n\|_F = \|\hat{S}_n A \hat{S}_n - \Lambda\|_F$, where Λ is a diagonal matrix. To conclude the proof, it is enough to observe that

$$\text{diag}(\hat{S}_n A \hat{S}_n) = \begin{bmatrix} A(1,1) & 0 & 0 \\ 0 & \text{diag}(S_{n-2} A(2:n-1, 2:n-1) S_{n-2}) & 0 \\ 0 & 0 & A(n,n) \end{bmatrix}.$$

and that $s(A) = S_n \text{diag}(S_n A S_n) S_n$. \square

To reduce the potential fluctuations in the coefficient of the elliptic operator in (1.5), we define a scaled anti-reflective BCs preconditioner for (1.5) by $D_M = D^{\frac{1}{2}} M D^{\frac{1}{2}}$, where M is defined in (3.5) and D is given in (3.2). Similarly, for the scaled equation in the form of (3.3), we give the unscaled anti-reflective BCs preconditioner by $M_D = \hat{H}^* \hat{H} + \alpha \hat{s}(\tilde{L}(u^k))$, where $\hat{H} = \hat{s}(H) \hat{s}(D^{-1/2})$. If Λ_H , Λ_D and $\Lambda_{\tilde{L}}$ denote the eigenvalue matrices of $\hat{s}(H)$, $\hat{s}(D^{-1/2})$ and $\hat{s}(\tilde{L}(u^k))$, respectively, then M_D can be written as

$$M_D = \hat{S}_n (\Lambda_H^* \Lambda_H \Lambda_D^* \Lambda_D + \alpha \Lambda_{\tilde{L}}) \hat{S}_n.$$

Next, we consider the preconditioned reblurring method. Let

$$(3.7) \quad \mathcal{AR} = \{T_n \Lambda T_n^{-1} : \Lambda \text{ is a real diagonal matrix of order } n\},$$

where T_n and T_n^{-1} are defined in (2.3). Unfortunately, $H \in \mathcal{AR}$ but $H^* \notin \mathcal{AR}$. However, in [12], it was proposed to use a reblurring approach, i.e., to replace H^* with H' , where H' is the matrix obtained by imposing anti-reflective BCs to the PSF rotated by 180 degrees. Since the PSF is assumed to be symmetric, $H' = H$ [13]. Therefore, instead of (1.5), one may solve the following equation [12]

$$(3.8) \quad A'_{u^k} u^{k+1} \equiv (H' H + \alpha L(u^k)) u^{k+1} = H' v, \quad k = 0, 1, \dots$$

by the Preconditioned BICGSTAB or PBiCGstab method [32] since A'_{u^k} is not symmetric. In this case, a reblurring preconditioner for (3.8) is given by

$$(3.9) \quad P = H' H + \alpha ar(L(u^k)).$$

Here, for any n -by- n matrix A , $ar(A)$ is defined by

$$ar(A) := \begin{bmatrix} z_1 + 2 \sum_{k=2}^{n-2} z_k & 0 & \dots & 0 & 0 \\ z_2 + 2 \sum_{k=3}^{n-2} z_k & & & & 0 \\ \vdots & & & & z_{n-2} \\ z_{n-3} + 2z_{n-2} & s(A(2:n-1, 2:n-1)) & & & z_{n-3} + 2z_{n-2} \\ & z_{n-2} & & & \vdots \\ 0 & & & & z_2 + 2 \sum_{k=3}^{n-2} z_k \\ 0 & 0 & \dots & 0 & z_1 + 2 \sum_{k=2}^{n-2} z_k \end{bmatrix},$$

where $z = (z_1, z_2, \dots, z_{n-2})^T$ is such that $s(A(2 : n-1, 2 : n-1)) = \mathcal{T}(z) - \mathcal{H}(\sigma^2(z), J\sigma^2(z))$. We only need form $s(A(2 : n-1, 2 : n-1))$ for computing $ar(A)$.

We note that $ar(A)$ belongs to the algebra \mathcal{AR} defined in (3.7)¹, where Λ is defined as in (2.5). Therefore, a linear system $Au = v$ can be solved within $O(n \log n)$ real operations by using three FSTs.

To reduce the potential fluctuations in the coefficient of the elliptic operator in (3.8), we define a scaled reblurring preconditioner for (3.8) as follows $D_P = D^{1/2}PD^{1/2}$, where P is defined in (3.9) and D is given in (3.2). For the scaled system

$$(3.10) \quad \tilde{A}'_{u^k} \tilde{u}^{k+1} \equiv \left(\tilde{H}' \tilde{H} + \alpha \tilde{L}(u^k) \right) \tilde{u}^{k+1} = \tilde{H}' v,$$

the unscaled reblurring preconditioned is defined by

$$P_D = ar(D^{-1/2}) H' H ar(D^{-1/2}) + \alpha ar(\tilde{L}(u^k)).$$

If Λ_H , Λ_D , and $\Lambda_{\tilde{L}}$ denote the eigenvalue matrices of H , $ar(D^{-1/2})$, and $ar(\tilde{L}(u^k))$, respectively, then P_D can be written as

$$P_D = T_n(\Lambda_H^* \Lambda_H \Lambda_D^* \Lambda_D + \alpha \Lambda_L) T_n^{-1}.$$

A further possibility is the use of anti-reflective BCs for $L(u^k)$. This implies that the coefficient matrix in (3.8) is closer to the preconditioner. Consequently, a faster convergence and a lower global cost have to be expected. This case is in fact considered in the numerics.

We comment on the cost of constructing unscaled preconditioners X_D , $X \in \{R, M, P\}$ and of each PCG/ PBiCGstab iteration. We note that $L(u^k)$ is a banded matrix. Therefore, computing $c(L(u^k))$, $\hat{s}(L(u^k))$, and $ar(L(u^k))$ needs only $O(n)$ operations [6, 7]. At each PCG/ PBiCGstab iteration, we need to calculate the matrix-vector product $\tilde{A}_{u^k} w$ and $\tilde{A}'_{u^k} w$ and solve the system $X_D y = b$. The vector multiplication $D^{-1/2} w$ can be computed in $O(n)$ operations since $D^{-1/2}$ is a diagonal matrix. $L(u^k) w$ can be done in $O(n)$ operations. For $H \in \mathcal{C}$ or $H \in \mathcal{AR}$, Hw , $H^* Hw$, and $H' Hw$ can be calculated in $O(n \log n)$ operations by two FCTs or FSTs plus computations of lower orders. The system $X_D y = b$ can also be solved in $O(n \log n)$ operations. Therefore, the total cost of each PCG/ PBiCGstab iteration is bounded by $O(n \log n)$, mainly due to four fast trigonometric transforms.

3.2. Two-dimensional problems. We can extend the results in Subsection 3.1 to image deblurring problems with different BCs. In the two-dimensional case, we assume that the PSF is quadrantly symmetric and normalized.

Let us consider the three algebras \mathcal{C} , $\hat{\tau}$, and \mathcal{AR} introduced in Section 3.1 together, and so we fix $t \in \{c, \hat{s}, ar\}$. For an n^2 -by- n^2 matrix A in the form of

$$(3.11) \quad A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{bmatrix},$$

¹In general, $ar(A) \neq \operatorname{argmin}_{B \in \mathcal{AR}} \|B - A\|_F$. Moreover, we can not construct $\operatorname{argmin}_{B \in \mathcal{AR}} \|B - A\|_F$ in only $O(n^2)$ operations by using the similar technique for computing $s(A)$ in [6].

where $A_{i,j}$ are n -by- n matrices, as defined in [8], the Level-1 transform preconditioner $t_1(A)$ is given by

$$t_1(A) = \begin{bmatrix} t(A_{1,1}) & t(A_{1,2}) & \cdots & t(A_{1,n}) \\ t(A_{2,1}) & t(A_{2,2}) & \cdots & t(A_{2,n}) \\ \vdots & \ddots & \ddots & \vdots \\ t(A_{n,1}) & t(A_{n,2}) & \cdots & t(A_{n,n}) \end{bmatrix},$$

and then the Level-2 transform preconditioner is

$$(3.12) \quad t_2(A) = Q t_1(Q^T t_1(A) Q) Q^T,$$

where Q is the permutation matrix which satisfies $[Q^T A Q]_{i,j;k,l} = [A]_{k,l;i,j}$ for $1 \leq i, j \leq n$ and $1 \leq k, l \leq n$, i.e., the (i, j) th entry of the (k, l) th block of A is permuted to the (k, l) th entry of the (i, j) th block. The expression of the Level-2 preconditioner in (3.12) is a classical result for $t = c$, while for $t \in \{\hat{s}, ar\}$ the same proof technique of Theorem 3.3 in [22] can be applied.

When one imposes the reflective BCs on H and the zero Neumann BCs on $L(u^k)$, the blurring matrix H is a block Toeplitz-plus-Hankel matrix with Toeplitz-plus-Hankel blocks, which can be diagonalized by the two-dimensional FCTs in $O(n^2 \log n)$ operations [22]. For the two-dimensional linear equation (1.5), using the fact that $c_2(H) = H$, we define the following optimal reflective BCs preconditioner

$$(3.13) \quad R = H^* H + \alpha c_2(L(u^k)).$$

To eliminate the possibility of large variations in the coefficient of the elliptic operator in (1.5), we employ the same strategy as in Section 3.1 by the diagonal scaling. Therefore, the scaled reflective BCs preconditioner is given by $D_R = D^{\frac{1}{2}} R D^{\frac{1}{2}}$, where R is defined in (3.13) and D is given in (3.2). Similarly, for the scaled system in (3.3), the unscaled reflective BCs preconditioner is given by $R_D = \hat{H}^* \hat{H} + \alpha c_2(\tilde{L}(u^k))$, where $\hat{H} = H c_2(D^{-1/2})$. Let Λ_H , Λ_D and $\Lambda_{\tilde{L}}$ denote the eigenvalue matrices of H , $c_2(D^{-1/2})$ and $c_2(\tilde{L}(u^k))$, respectively. Then R_D can be written as

$$R_D = (C_n \otimes C_n)^T (\Lambda_H^* \Lambda_H \Lambda_D^* \Lambda_D + \alpha \Lambda_{\tilde{L}}) (C_n \otimes C_n),$$

and hence it is easily inverted by employing few FCTs in $O(n^2 \log n)$ operations.

Next, we assume the anti-reflective BCs for H and the Neumann BCs for $L(u^k)$. Then, we design the sine-based transform preconditioner for (1.5) by

$$M = \hat{s}_2(H)^* \hat{s}_2(H) + \alpha \hat{s}_2(L(u^k)).$$

By employing the diagonal scaling, we define the scaled anti-reflective BCs preconditioner $D_M = D^{1/2} M D^{1/2}$ for (1.5) and the unscaled anti-reflective BCs preconditioner $M_D = \hat{H}^* \hat{H} + \alpha c_2(\tilde{L}(u^k))$, where $\hat{H} = \hat{s}_2(H) \hat{s}_2(D^{-1/2})$, for the scaled two-dimensional system (3.3). Let Λ_H , Λ_D and $\Lambda_{\tilde{L}}$ denote the eigenvalue matrices of H , $\hat{s}_2(D^{-1/2})$ and $\hat{s}_2(\tilde{L}(u^k))$, respectively. Then M_D takes the form

$$M_D = (\hat{S}_n \otimes \hat{S}_n) (\Lambda_H^* \Lambda_H \Lambda_D^* \Lambda_D + \alpha \Lambda_{\tilde{L}}) (\hat{S}_n \otimes \hat{S}_n),$$

which is computationally attractive via FSTs since any matrix operation can be done within $O(n^2 \log n)$ operations.

Finally, we design the reblurring preconditioner $ar_2(A'_{u^k})$ for (3.8) assuming the anti-reflective BCs for H and the Neumann BCs or the anti-reflective BCs for $L(u^k)$. Since H is the anti-reflective BCs matrix, we define a reblurring preconditioner for (3.8) as $P = H'H + \alpha ar_2(L(u^k))$. Also, we propose the scaled reblurring preconditioner $D_P = D^{1/2}PD^{1/2}$ for (3.8) and the unscaled reblurring preconditioner $P_D = \tilde{H}'\tilde{H} + \alpha ar_2(\tilde{L}(u^k))$ for the two-dimensional scaled linear system (3.10), where $\tilde{H} = H \cdot ar_2(D^{-1/2})$. Let Λ_H , Λ_D , and $\Lambda_{\tilde{L}}$ denote the eigenvalue matrices of H , $ar_2(D^{-1/2})$, and $ar_2(\tilde{L}(u^k))$, respectively. Then, the two-dimensional preconditioner P_D can be written as

$$P_D = (T_n \otimes T_n)(\Lambda_H^* \Lambda_H \Lambda_D^* \Lambda_D + \alpha \Lambda_{\tilde{L}})(T_n \otimes T_n)^{-1}.$$

Again, these two-dimensional preconditioners show interesting computational features since the associated linear systems can be solved within $O(n^2 \log n)$ operations.

4. Asymptotic spectral analysis of the preconditioned sequences. To study the effectiveness of the proposed preconditioners, we need the clustering analysis of the spectrum. Also, localization of eigenvalues is of interest when solving (3.3) via PCG or (3.10) by PBiCGstab [1]. Here is a useful definition [28] for sequences of matrices $\{A_n\}$ where A_n has size d_n , n positive integer, and $d_k > d_q$ if $k > q$.

DEFINITION 4.1. *A matrix sequence $\{A_n\}$ is said to be distributed (in the sense of the eigenvalues) as the pair (θ, G) , or has the distribution function θ , if, for any $F \in \mathcal{C}_0(\mathbb{C})$, the following limit relation holds*

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n F(\lambda_j(A_n)) = \frac{1}{\mu(G)} \int_G F(\theta(t)) dt, \quad t = (t_1, \dots, t_d),$$

where $\{\lambda_j(A_n)\}_{j=1}^n$ denote the eigenvalues of A_n and $\mu(\cdot)$ is the standard Lebesgue measure. In that case we write $\{A_n\} \sim_\lambda (\theta, G)$.

An interesting consequence of the equation (4.1) is that $\{A_n\} \sim_\lambda (\theta, G)$ implies that most of the eigenvalues are contained within any ϵ -neighborhood of the essential range of θ . That is, the range of θ is a cluster for the spectrum of $\{A_n\}$.

The main observation is that all the matrices considered so far are low-rank perturbations of Toeplitz matrices or can be viewed as extracted from Generalized Locally Toeplitz (GLT) sequences (see [28] and references therein and the seminal work [31]). We note that every GLT sequence has a symbol and this symbol is the spectral distribution function. Furthermore, the class of GLT sequences is an algebra of matrix-sequences. Hence, when making linear combinations, products, or inverses (when the symbol does not vanish on sets of positive measure), the result is a new GLT sequence whose symbol can be obtained via the same operations on the original symbols. Therefore, the preconditioned matrices can be seen again as extracted from a GLT sequence whose symbol is the ratio of the symbols: here the numerator is the symbol of the original matrix sequence and the denominator is the symbol of the preconditioning sequence.

In this section, according to Definition 4.1 and since we are interested in asymptotic estimates, we are forced to indicate explicitly the parameter n which uniquely defines the size of the associated matrix. First, we discuss in detail the case of reflective BCs. When considering $B_n = L(u^k)$, it is known that [28]

$$\begin{aligned} \{B_n\} &\sim_\lambda (a(x)w(t), G), & G &= \Omega \times [0, 2\pi]^d, \\ a(x) &= \frac{1}{\sqrt{|\nabla x|^2 + \beta^2}} & w(t) &= \sum_{i=1}^d (2 - 2 \cos(t_i)). \end{aligned}$$

On the other hand, $c(B_n) \sim_\lambda (\bar{a}w(t), G)$, where \bar{a} is a constant and in fact it is the mean of the function $a(x)$: $\bar{a} = \frac{1}{\mu(\Omega)} \int_\Omega a(x) dx$. The sequence $\{c(B_n)^{-1}B_n\}$ is clustered at one only if the sequence $\{B_n - c(B_n)\}$ is clustered at zero. Since $\{B_n - c(B_n)\} \sim_\lambda ((a(x) - \bar{a})w(t), G)$, the optimal cosine preconditioner is effective only if the function $a(x)$ has no large variation. To obtain a clustering preconditioner, a diagonal scaling has to be introduced. The preconditioner $\text{diag}(B_n)^{1/2}c(B_n)\text{diag}(B_n)^{1/2}$ is such that $\{\text{diag}(B_n)^{1/2}c(B_n)\text{diag}(B_n)^{1/2}\} \sim_\lambda (a(x)w(t), G)$ due to the algebra structure of GLT sequences. Hence, the preconditioned sequence is clustered at one.

In our case, the coefficient matrix $A_n = H^*H + \alpha L(u^k)$ is the sum of an integral approximate operator and an approximate elliptic differential operator. We note that $\{A_n\} \sim_\lambda (|\hat{h}(t)|^2 + \alpha a(x)w(t), G)$, where \hat{h} is the symbol of the PSF defined in (2.4) for the 1D case and similarly can be defined for $d > 1$ (the entries of the PSF are the Fourier coefficients of \hat{h}). An effective preconditioner has to consider both terms which constitute the matrix A_n . This is the aim of the preconditioner R_n defined in (3.1) and (3.13) for the 1D and 2D case, respectively. We have $\{R_n\} \sim_\lambda (|\hat{h}(t)|^2 + \alpha \bar{a}w(t), G)$ and so $\{A_n - R_n\} \sim_\lambda ((a(x) - \bar{a})\alpha w(t), G)$. In this case, we can not apply a diagonal scaling to $c(L(u^k))$ because otherwise we lose the computational efficiency, the matrix $H^*H + \alpha \text{diag}(L(u^k))^{1/2}c(L(u^k))\text{diag}(L(u^k))^{1/2}$ can not be diagonalized by discrete cosine transforms. Therefore, we have to apply to R_n a diagonal scaling which should act like $\text{diag}(L(u^k))^{1/2}$ on $c(L(u^k))$, while it should not affect the term H^*H .

Unfortunately, since we have a diagonal scaling we can not apply the diagonal scaling only to a term of the sum. To balance the contribution of the two terms, the diagonal scaling is defined by the matrix D (defined in (3.2)) which leads to the scaled preconditioner D_R . We have $\{(D_R)_n\} \sim_\lambda ((1 + \alpha a(x))(|\hat{h}(t)|^2 + \alpha w(t)), G)$ and hence the preconditioned sequence is not clustered at one, even if for values of α used in the considered applications it shows an optimal behaviour (see Figure 5.2).

We recall that the clustering is a useful property but it is not strictly necessary for the optimality of the related preconditioned Krylov method: for instance in the Hermitian positive definite case and when dealing with the PCG iterations, the spectral equivalence is sufficient. Since $D_R^{-1}A_n$ is similar to $R^{-1}\tilde{A}_n$, with $\tilde{A}_n = D^{-1/2}A_nD^{-1/2}$, the use of the scaled preconditioner D_R to the linear system (1.5) is equivalent to applying the preconditioner R to the scaled linear system (3.3). However, the scaled preconditioner D_R is more effective than R for large values of α (see numerical results in Section 5).

Remark 4.2. For small values of α , i.e., when few regularization is required, the three preconditioners R , D_R and R_D have a similar behaviour. Moreover, when α goes to zero the effectiveness of the proposed preconditioners increases because the preconditioners and the original coefficient matrix A_n all tend to H^*H .

To conclude this section, we note that in the case of anti-reflective BCs similar considerations can be done. The main difference is when we consider the reblurring strategy. However, using the results in [15], the nonsymmetric case can be considered as well since the antisymmetric part has trace norm (sum of all singular values) bounded by a pure constant independent of n . Therefore the spectral distribution is governed by the symmetric part which is dominant as discussed in Section 3.3 of [3].

5. Numerical tests. We solve problem (1.4) by the FP method (1.5) with the operator \mathcal{H} approximated by using different BCs and with the operator \mathcal{L}_u imposed by zero Neumann BCs or anti-reflective BCs. The algorithm was implemented in MATLAB 7.10 and run on a PC Intel Pentium IV of 3.00 GHZ CPU. We shall show the effectiveness of the proposed preconditioners for signal/image deblurring and also

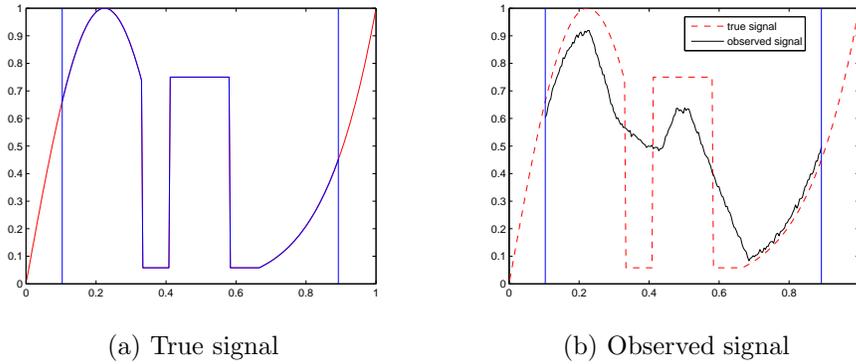


FIG. 5.1. True and observed signals

give a comparison of the quality of the restored signals/images with different BCs.

In our test, we choose initial guess $u^0 = v$ for the FP algorithm. We shall solve (1.5) by the PCG method when the Neumann BCs imposed on $L(u^k)$ and solve (1.5) by the PBiCGstab method when the anti-reflective BCs imposed on $L(u^k)$. Also, we solve (3.8) by the PBiCGstab method. The initial guess for the PCG and PBiCGstab methods in k th FP iteration is chosen to be the $(k-1)$ th FP iterate. The inner PCG and PBiCGstab iterations are stopped when the residual vector r_k of the linear systems (1.5) and (3.8) at the k th iteration satisfies $\|r_k\|/\|r_0\| < tol$, where $\|\cdot\|$ denotes the 2-norm and tol is set to 10^{-6} and 10^{-5} in the 1D and 2D case, respectively.

5.1. 1D case: Signal deblurring. In our experiments, we suppose the true signal u is given as in Figure 5.1(a). The two vertical lines shown in Figure 5.1(a) denote the field of view (i.e., $[0.1, 0.9]$) of our signal and the signal outside the two vertical lines can be approximated by different BCs. The true signal is blurred by the symmetric out of focus PSF:

$$(5.1) \quad h_i = \begin{cases} c & \text{if } |i| < m(n), \\ 0 & \text{otherwise,} \end{cases}$$

where c is the normalization constant such that $\sum_i h_i = 1$ and $m(n)$ is the center of the PSF which depends on n so that the restored signal lies in the interval $[0.1, 0.9]$. A Gaussian noise η with a specific noise-to-signal ratio $\|\eta\|/\|Hu\|$ is added to the blurred signal. We consider the true signal is blurred by the out of focus PSF and then added the Gaussian noise with the noise levels 1%, i.e., $\|\eta\|/\|Hu\| = 0.01$. Figure 5.1(b) show the observed signal.

We now show that the proposed preconditioners are effective for solving (1.5) and (3.8) with different BCs. In our numerical experiments, the FP iteration is stopped when $\|u^k - u^{k-1}\|/\|u^k\| < 10^{-3}$. We will concentrate on the performance of different choices of preconditioners for various values of the regularization parameters α and β in (1.3) and the size n of the coefficient matrix A_{u^k} in (1.5).

In Tables 1 and 2, we report the average number of iterations per FP iteration. According to Remark 4.2, the effectiveness of the proposed preconditioners increases when α decreases. Moreover, decreasing α all the proposed preconditioners become equivalent, explicitly the PCG/PBiCGstat converges in about the same number of iterations.

PCG	Reflective						AR+Sine+ZN L					
α	N	I	D	R	D_R	R_D	N	I	D	M	D_M	M_D
10^{-1}	30	269	163	73	49	45	28	221	155	60	51	36
10^{-2}	37	172	107	84	37	32	24	149	94	67	33	25
10^{-3}	32	99	71	63	42	36	24	80	59	56	37	31
10^{-4}	19	57	56	38	36	35	19	60	58	31	30	29
10^{-5}	20	74	71	23	25	24	17	47	45	20	20	19
10^{-6}	11	122	122	7	7	8	8	85	86	14	14	14

PBiCGstab	AR+Reblur+ZN L						AR+Reblur+AR L					
α	N	I	D	P	D_P	P_D	N	I	D	P	D_P	P_D
10^{-1}	27	178	105	57	53	51	24	179	109	41	46	35
10^{-2}	24	105	59	59	25	21	23	107	60	39	23	20
10^{-3}	23	60	44	43	27	23	22	54	37	34	22	20
10^{-4}	20	33	31	20	18	18	20	32	30	18	16	17
10^{-5}	25	31	33	10	9	10	20	31	32	10	10	10
10^{-6}	9	78	68	4	4	4	11	56	62	4	4	4

TABLE 1

Average number of PCG/PBiCGstab iterations per FP step varying α , with $n = 203$ and $\beta = 0.1$. Here, N , I , D , “Reflective”, “AR+Sine+ZN L”, “AR+Reblur+ZN L”, and “AR+Reblur+AR L” denote the number of FP steps, no preconditioner and the diagonal scaling preconditioner, the reflective BCs, the anti-reflective BCs with sine preconditioner by imposing the zero Neumann BCs on $L(u^k)$, the anti-reflective BCs with reblurring by imposing the zero Neumann BCs on $L(u^k)$, and the anti-reflective BCs with reblurring by imposing the anti-reflective BCs on $L(u^k)$, respectively.

PCG	Reflective						AR+Sine+ZN L					
β	N	I	D	R	D_R	R_D	N	I	D	M	D_M	M_D
10^{-3}	31	434	245	305	298	125	24	349	200	297	226	85
10^{-2}	31	218	139	149	98	67	24	175	112	139	72	46
10^{-1}	32	99	71	63	42	36	24	80	59	56	37	31
10^0	28	39	36	21	18	20	21	35	32	19	16	15

PBiCGstab	AR+Reblur+ZN L						AR+Reblur+AR L					
β	N	I	D	P	D_P	P_D	N	I	D	P	D_P	P_D
10^{-3}	23	403	254	361	346	59	22	323	207	288	285	57
10^{-2}	23	157	87	130	58	43	22	134	77	102	43	28
10^{-1}	23	60	44	43	27	23	22	54	37	34	22	20
10^0	20	21	20	12	11	11	20	19	18	10	9	9

TABLE 2

Average number of PCG/PBiCGstab iterations per FP step varying β , with $n = 203$ and $\alpha = 0.001$.

We note that anti-reflective BCs usually require fewer steps and fewer PCG / BiCGstab iterations per FP step when compared with reflective BCs. This shows that the improvement in the model also leads to an improvement in the global computational complexity of the numerical methods. This is more evident for the optimal restoration since the anti-reflective BCs require a regularization parameter α smaller than that of reflective BCs (Figures 5.6(a) and 5.4).

Figure 5.2 describes the average PCG/PBiCGstab iterations per FP step varying n . We note that the preconditioners with a diagonal scaling show the best behavior.

In Figure 5.3, we present the restored signals for varying β , e.g., by solving (1.5) when anti-reflective BCs have been imposed. As expected, the recovered signals become sharper when the value of β is smaller, the value $\beta = 0.1$ gives a restoration

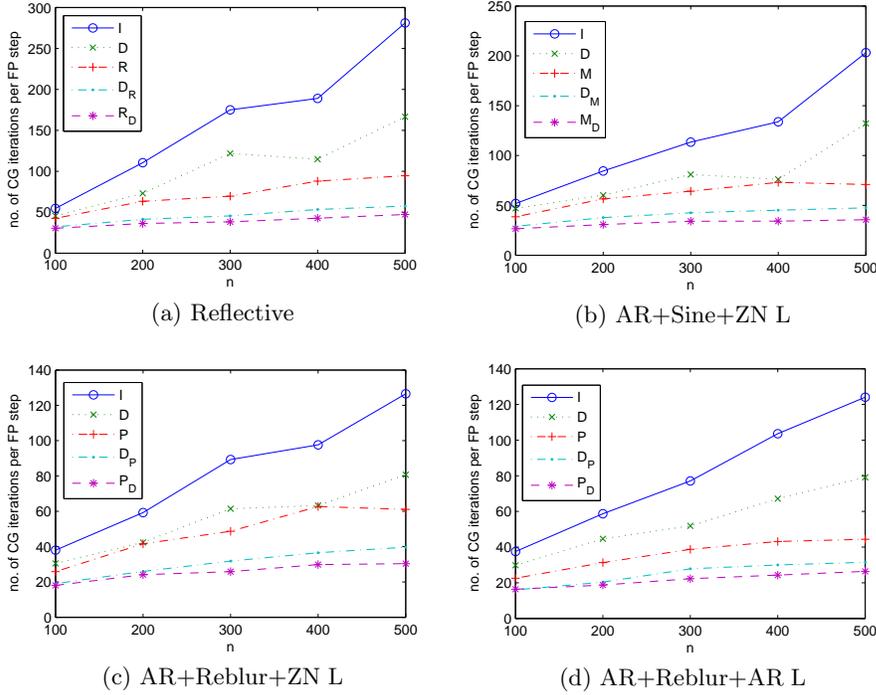


FIG. 5.2. Average number of PCG/PBiCGstab iterations per FP step for various n , with $\alpha = 10^{-3}$ and $\beta = 0.1$.

sufficiently good anyway.

We can easily observe from Tables 1-2 and Figure 5.2 that the proposed preconditioners with a diagonal scaling are the most effective preconditioner when varying parameters α , β , and n . Finally, we remark that in all our tests, the proposed algorithm needs the same number of FP steps for the no-preconditioner and preconditioned cases and $\|g(u^k)\|$ (see (1.4) for the definition of $g(\cdot)$), tends to $O(10^{-5})$ or $O(10^{-6})$ at the final FP iterate.

To check the quality of restored signals by using different BCs, in Figure 5.6(a) we show the relative restoration error (RRE), $\|u_\alpha - u\|/\|u\|$, where u_α is the computed approximation of the true signal u , versus the regularization parameter α . Figure 5.4 gives the restored signals with optimal value of the parameter α , where α_{opt} , Re. , Fp. , and It. denote the optimal α , the minimal RRE, the number of FP steps, and the average number of PCG/BiCGstab iterations per FP step, respectively.

From Figure 5.4 we argue that anti-reflective BCs lead to the most accurate restored signals with less significant ringing effects at the edges and fewer PCG iterations per FP step, when compared with reflective BCs. Moreover, thanks to the improvement in the model of the problem, anti-reflective BCs require less regularization than reflective BCs. This implies a smaller α_{opt} and hence a small number of PCG/BiCGstab iterations per FP step, while the number of FP iterations remains about the same.

5.2. 2D case: Image deblurring. In this section, we apply the proposed preconditioners to image restoration with different BCs. Suppose the true images are blurred by the Gaussian blur and then suppose that a white Gaussian noise η with

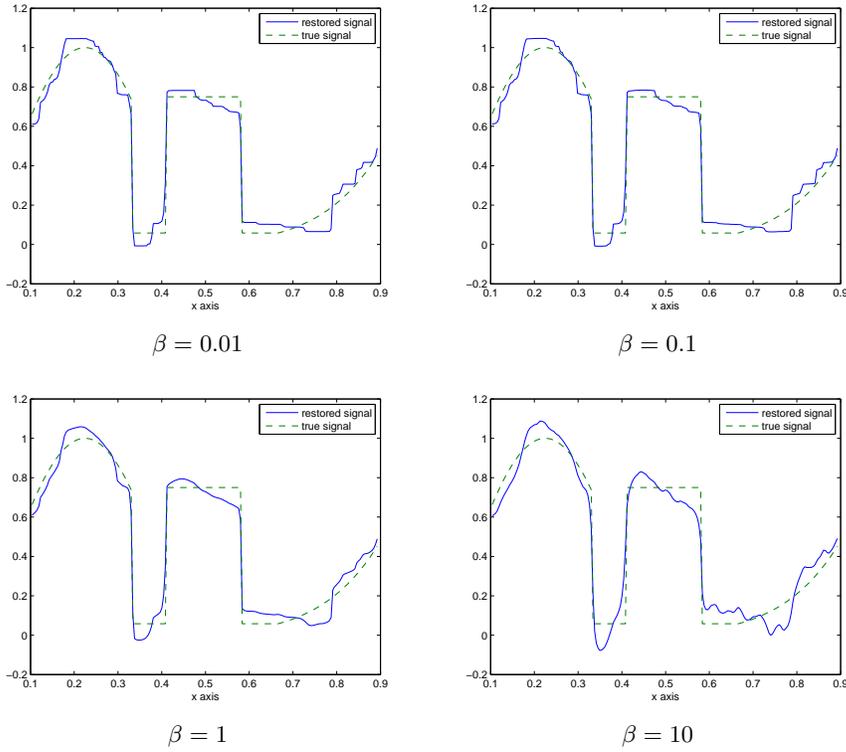


FIG. 5.3. Restorations for Anti-Reflective BCs based (1.5) with $n = 203$, $\alpha = 10^{-3}$ varying β .

PCG	Reflective						AR+Sine+ZN L					
	N	I	N	D_R	N	R_D	N	I	N	D_M	N	M_D
10^0	60	244	60	48	60	23	60	229	60	210	61	23
10^{-1}	41	129	46	29	33	22	40	114	40	117	28	38
10^{-2}	13	70	13	44	11	39	12	58	9	49	8	41
10^{-3}	3	124	2	42	2	42	2	147	2	36	2	37
10^{-4}	2	164	1	23	1	23	1	354	1	23	1	23

PBiCGstab	AR+Reblur+ZN L						AR+Reblur+AR L					
	N	I	N	D_P	N	P_D	N	I	N	D_P	N	P_D
10^0	58	52	*	*	59	13	60	52	59	50	60	7
10^{-1}	27	27	33	16	31	7	26	28	31	9	30	5
10^{-2}	10	9	10	7	10	6	10	10	12	5	10	5
10^{-3}	2	16	2	5	2	5	3	11	2	4	2	5
10^{-4}	1	33	1	3	2	1	1	34	1	3	2	1

TABLE 3

Average number of CG/BiCGstab iterations per FP step for varying α . Here, $\beta = 0.01$ (* means that the method does not converge).

the noise levels 0.1% is added. Figure 5.5 shows the true and observed images.

In our numerical tests, the FP iteration is stopped when $\|u^k - u^{k-1}\|/\|u^k\| < 10^{-4}$ and the maximal number of FP steps is set to be 100. We fix $\beta = 0.01$ and only focus on the performance of different choices of preconditioners for varying α .

In Table 3 the number of iterations is displayed for solving (1.5) or (3.8) with

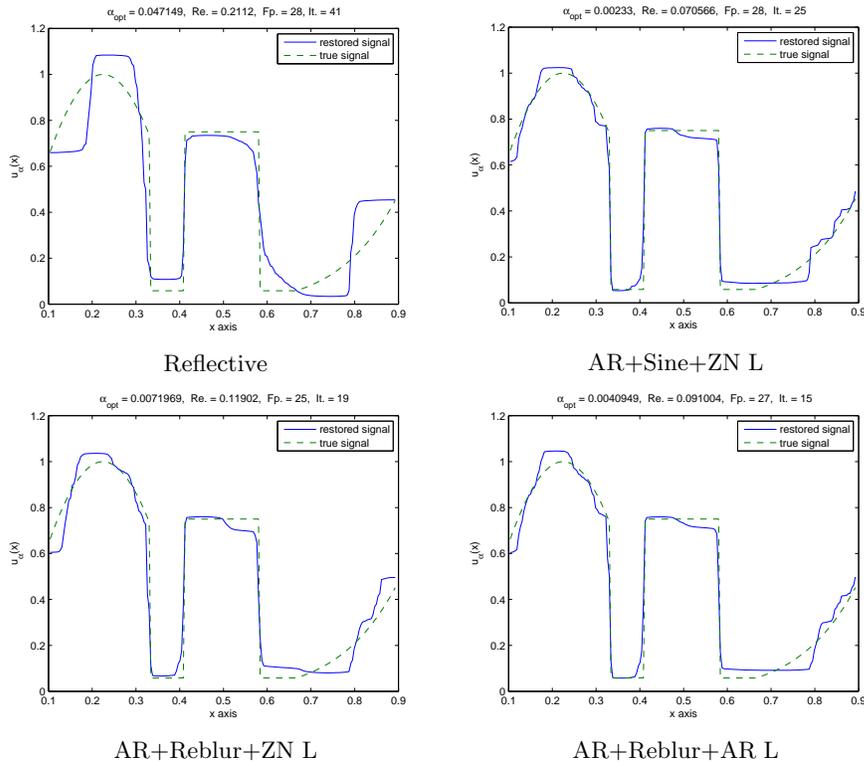


FIG. 5.4. Restored signals with different BCs. Here $n = 203$ and $\beta = 0.1$



FIG. 5.5. True and observed images

different BCs and various values of α , where N and I mean the number of FP iterations and no preconditioner, respectively. Here, we only give the average number of CG/BiCGstab iterations per FP step. Table 3 suggests that the unscaled preconditioners X_D , with $X \in \{R, M, P\}$, are very effective matrix approximations for all values of α , while the scaled preconditioners D_X do not work well for large values of α especially for anti-reflective BCs. However, concerning anti-reflective BCs a good choice for α is in the interval $[10^{-3}, 10^{-2}]$ and in this case both choices X_D and D_X have a similar behaviour. We note that the number of FP iterations decrease with

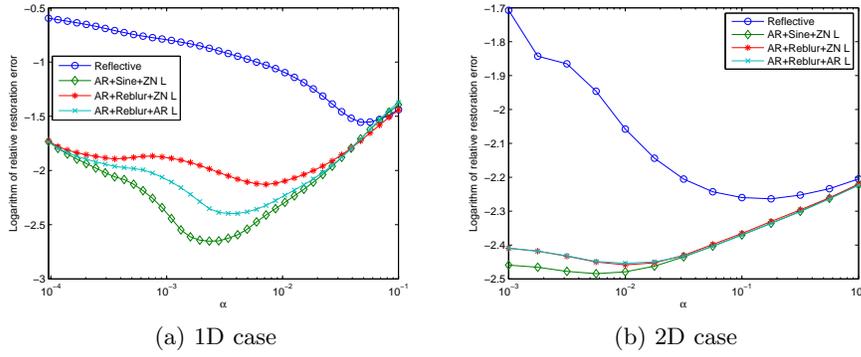


FIG. 5.6. The RRE versus the regularization parameter α for different BCs.

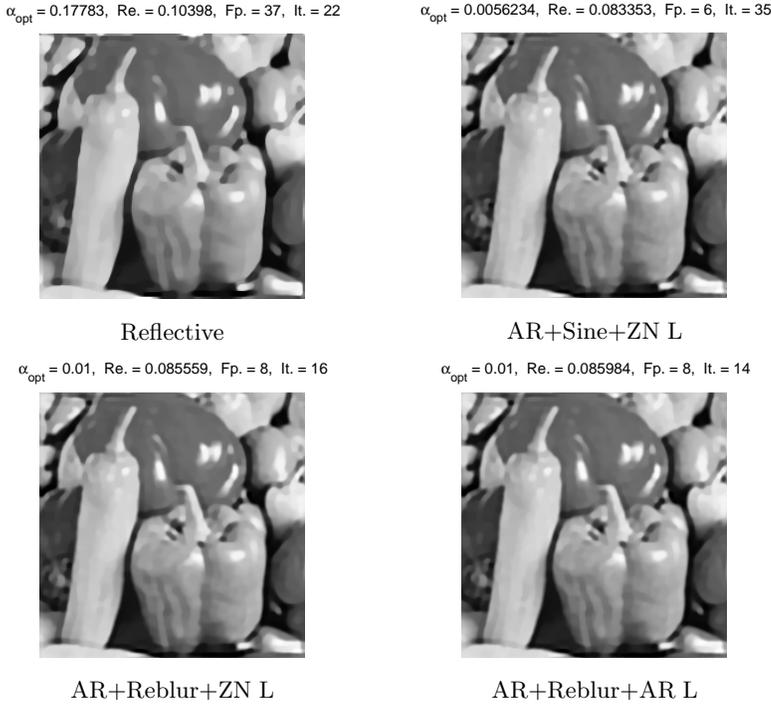


FIG. 5.7. Restored images with different BCs.

α , so if we have a good model that requires a lower regularization we obtain a gain also in terms of the computational cost of the whole restoration procedure. In all our tests, it is shown that $\|g(u^k)\|$ tends to $O(10^{-3})$ or $O(10^{-4})$ at the final FP iterate.

Next, we check the quality of restored images by using different BCs. Figure 5.6(b) describes the relative restoration error (RRE) $\|u_\alpha - u\|/\|u\|$, where u_α is the computed approximation of the true image u , versus the regularization parameter α . Figure 5.7 presents the restored images with optimal value of the parameter α . Like in the 1D case, Figure 5.7 shows that the anti-reflective BCs lead to better restored images and sharper edges with a lower computational cost than the reflective BCs (see the high reduction in the FP iterations with smaller α_{opt}).

6. Conclusions. In this paper, we have considered the effect of reflective and anti-reflective BCs, when regularizing blurred and noisy images via the total variation approach. In particular, we have studied some preconditioning strategies for the linear systems arising from the FP iteration given in [33]. In the case of anti-reflective BCs we have also considered a comparison with the reblurring idea proposed in [13]. We recall that the reblurring method has been shown effective when combined with the Tikhonov regularization and here one of the issues was to verify that reblurring and total variation can be combined satisfactorily. Furthermore, the optimal behavior of our preconditioners has been validated numerically.

Besides the computational features of the preconditioning techniques, we stress the improvement obtained via anti-reflective BCs both in terms of reconstruction quality and reduction of the computational cost. In fact, the precision of such BCs was already known in the relevant literature (see [11, 12, 24] and the references therein). However, this is the first time that the anti-reflective BCs have been combined with a sophisticated regularization method, where the use of fast transforms is very welcome for saving computational cost. The anti-reflective BCs require a smaller regularization parameter α than that of other BCs (see numerical results in Section 5). Also, the proposed preconditioners are more effective when α becomes smaller. All these features lead to an efficient method that converges within few iterations and that requires only four fast trigonometric transforms plus lower order computations. Therefore, the total complexity could be comparable with that of the efficient FTVd method proposed in [34]: in this respect, as a future line of research, a modification of the FTVd method including the anti-reflective BCs has to be investigated.

Besides deconvolution problems, the proposed anti-reflective BCs and the preconditioning strategies for the lagged diffusivity algorithm in (1.5) could be also applied to more general forward operators, where symmetric shift-invariant operators should be considered.

Potential lines of interest for future investigations could include the use of anti-reflective BCs in promising split Bregman method [14] and FTVd [34], and a more precise clustering analysis of the preconditioning sequences, in the spirit of Section 4: in particular, according to the analysis in [28], it is important to concentrate the efforts in new preconditioner design in such a way that the resulting preconditioned sequence is a GLT sequence showing a symbol concentrated at 1 as much as possible.

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