On the sufficient and necessary condition for the inverse problem of a mass-spring system

Zheng-Jian Bai [1] +

[1] Department of Information and Computational Mathematics, Xiamen University

Xiamen 361005, People's Republic of China

Abstract. In this paper, we consider the inverse problem for a special mass-spring system from its one or two eigenpairs, where the mass and stiffness matrices are symmetric positive definite with the mass matrix being diagonal and the stiffness matrix being tridiagonal. The sufficient and necessary condition on the prescribed eigendata for the existence of a physical solution is provided.

Keywords: Mass-spring system; inverse eigenvalue problem; quadratic pencil

1. Introduction

In this paper, we consider the inverse problem of the reconstruction of a nontrivial quadratic pencil $Q(\lambda) := \lambda^2 M + K$ from its eigendata, where the symmetric positive definite matrices *M* and *K* are given by

$$M = \mathrm{diag}(m_1, \ldots, m_n) \text{ and } K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_n & -k_n \end{pmatrix}. \quad (1)$$

This inverse problem arises in vibration, see [3, 5]. In practical applications, the matrices *M* and *K* are called the mass and stiffness matrices, respectively, and the reconstructed model should be physical, i.e., the mass and stiffness parameters m_j and k_j must be positive. Moreover, the problem will be over-determined if given more than three eigenpairs, which should be avoided. Therefore, we focus on the inverse problem stated as follows:

Problem A1. Given one prescribed pair $(i\omega, z)$, where $i := \sqrt{-1}, 0 \neq \omega \in \mathbb{R}, 0 \neq z \in \mathbb{R}^n$. Find the positive parameters m_j and k_j such that $(i\omega, z)$ is an eigenpair of the corresponding quadratic pencil $Q(\lambda)$.

Problem A2. Given two prescribed pairs $(i\alpha, x)$ and $(i\beta, y)$, where $0 \neq \alpha, \beta \in \mathbb{R}, 0 \neq x, y \in \mathbb{R}^n$. Find the positive parameters m_j and k_j such that $(i\alpha, x)$ and $(i\beta, y)$ are two eigenpairs of the corresponding quadratic pencil $Q(\lambda)$.

In this paper, we will discuss the solvability of Problems A1 and A2. This is motivated by the recent paper by Chu, Buono, and Yu [2]. Chu, Buono, and Yu [2] gave the necessary condition for the construction of the pencil $Q(\lambda)$ from one eigenpair. We provide the sufficient and necessary condition for the solvability of Problems A1 and A2. In particular, in Problem A1, the stiffness parameters are used as the free parameters and the mass parameters are expressed in terms of the stiffness parameters. In Problem A2, we choose the stiffness parameter k_n as the free parameter and express the mass and stiffness parameters and the remaining stiffness parameters in terms of k_n. This is more applicable in

+ Corresponding author. Tel.: +86-592-2580 687; fax: +86-592-2580 608.
E-mail address: zjbai@xmu.edu.cn

certain applications as noted in [2]. We also point out that the solution to Problem A2 is unique if the total mass is given.

2. Solvability of Problem A1

Since $(i\omega, z)$ is an eigenpair of $Q(\lambda)$, we have

$$(K-\omega^2 M)z = 0. \tag{2}$$

By (1), (2) can be rewritten as the following *n* equations:

$$\omega^2 z_n m_n = d_n k_n,\ \ \omega^2 z_j m_j = d_j k_j - d_{j+1}k_{j+1} \text{ for } j = n-1,\ldots,2, \text{ and } \omega^2 z_1 m_1 = z_1 k_1 - d_2 k_2, \tag{3}$$

where $d_j = z_j - z_{j-1}$ for $j = n,\ldots,2$. Now, we establish the sufficient and necessary condition on the data $(i\omega, z)$ for the construction of positive m_j and k_j successively for $j = n,\ldots,1$.

1) The existence of $m_n > 0$ and $k_n > 0$. It is easily follows that $z_n \neq 0$. Otherwise, by (3), we obtain that $z = 0$. This is a contradiction since $z \neq 0$. Thus $z_n \neq 0$. By (3), there exist $m_n > 0$ and $k_n > 0$ if and only if $z_n d_n > 0$. In this case, $k_n > 0$ is arbitrary and m_n is determined by $m_n = d_n/(\omega^2 z_n)k_n$.

2) The existence of $m_j > 0$ and $k_j > 0$ for $j = n,\ldots,2$. We consider the existence problem from the following three cases:

a) $z_j = 0$. In this case, $m_j > 0$ is arbitrary, and k_j satisfies $z_{j-1}k_j = -z_{j+1}k_{j+1}$ for some $k_{j+1} > 0$. We have $z_{j-1} \neq 0$. Otherwise, it follows that $z_n = 0$, which contradicts that $z_n \neq 0$. Therefore, there exists $k_n > 0$ if and only if $z_{j-1}z_{j+1} < 0$. In this case, $k_j = -z_{j+1}/z_{j-1}k_{j+1}$.

b) $z_j > 0$. By (3), there exists $m_j > 0$ if and only if $d_j k_j > d_{j+1}k_{j+1}$ for some $k_{j+1} > 0$. Next, we discuss the existence of $k_j > 0$ from three cases as follows:

i. $d_j > 0$. Then $k_j > \max\{0, d_{j+1}/d_j k_{j+1}\}$.

ii. $d_j = 0$. Then there exists $k_j > 0$ if and only if $d_{j+1} < 0$. In this case, $k_j > 0$ is arbitrary.

iii. $d_j < 0$. Then there exists $k_j > 0$ if and only if $d_{j+1} < 0$. In this case, $0 < k_j < d_{j+1}/d_j k_{j+1}$.

c) $z_j < 0$. By (3), there exists $m_j > 0$ if and only if $d_j k_j < d_{j+1}k_{j+1}$ for some $k_{j+1} > 0$. Next, we discuss the existence of $k_j > 0$ from the following three cases:

i. $d_j > 0$. Then there exists $k_j > 0$ if and only if $d_{j+1} > 0$. In this case, $0 < k_j < d_{j+1}/d_j k_{j+1}$.

ii. $d_j = 0$. Then there exists $k_j > 0$ if and only if $d_{j+1} > 0$. In this case, $k_j > 0$ is arbitrary.

iii. $d_j < 0$. Then $k_j > \max\{0, d_{j+1}/d_j k_{j+1}\}$.

3) The existence of $m_1 > 0$ and $k_1 > 0$. We claim that $z_1 \neq 0$ (Otherwise, we can get $z_n = 0$. This is a contradiction). Then, we can discuss the problem from the following two cases:

a) $z_1 > 0$. By (3), there exists $m_1 > 0$ if and only if $z_1 k_1 > d_2 k_2$ for some $k_2 > 0$. This holds if and only if $k_1 > \max\{0, d_2 / z_1 k_2\}$. Then $m_1 = (z_1 k_1 - d_2 k_2)/(\omega^2 z_1)$.

b) $z_1 < 0$. By (3), there exists $m_1 > 0$ if and only if $z_1 k_1 < d_2 k_2$ for some $k_2 > 0$. This holds if and only if $k_1 > \max\{0, d_2 / z_1 k_2\}$. Then $m_1 = (z_1 k_1 - d_2 k_2)/(\omega^2 z_1)$.

Based on the above analysis, we obtain the following result on the solvability of Problem A1.

Theorem 1. Suppose that ω is a real nonzero scalar and z is a real nonzero n-vector. Let $d_j = z_j - z_{j-1}$ for $j = n, \ldots, 2$. Then Problem A1 is solvable if and only if z satisfies the following conditions successively.

1) $z_n d_n > 0$**.** In this case, $k_n > 0$ is arbitrary and m_n is determined by $m_n = d_n / (\omega^2 z_n) k_n$.

2) For $j = n, \ldots, 2$,

a) $z_j = 0$ and $z_{j-1} z_{j+1} < 0$**.** In this case, $k_j = -z_{j+1} / z_{j-1} k_{j+1}$ for some $k_{j+1} > 0$ and $m_j > 0$ is arbitrary or

b) $d_j = 0$ and $z_j d_{j+1} < 0$**.** In this case, $k_j > 0$ is arbitrary and $m_j > 0$ is given by $m_j = -d_{j+1} / (\omega^2 z_j) k_{j+1}$ for some $k_{j+1} > 0$ or

c) $z_j d_j > 0$ **.** In this case, $k_j > \max\{0, d_{j+1} / d_j k_{j+1}\}$ and $m_j = (d_j k_j - d_{j+1} k_{j+1}) / (\omega^2 z_j)$ for some $k_{j+1} > 0$ or

d) $z_j d_j < 0$ and $z_j d_{j+1} < 0$ **.** In this case, $0 < k_j < d_{j+1} / d_j k_{j+1}$ and $m_j = (d_j k_j - d_{j+1} k_{j+1}) / (\omega^2 z_j)$ for some $k_{j+1} > 0$.

3) $z_1 \neq 0$. In this case, $k_1 > \max\{0, d_2 / z_1 k_2\}$ and $m_1 = (z_1 k_1 - d_2 k_2)/(\omega^2 z_1)$ for some $k_2 > 0$.

Let m_j^o and k_j^o be a-priori estimate of the unknown parameters m_j and k_j. It is desirable to find the optimal parameters m_j and k_j by solving the optimal parameter updating problems successively:

$$\min \left\| k_n - k_n^o \right\| \text{ subject to } k_n \geq \varepsilon \quad , \tag{4}$$

$$\min \left\| k_j - k_j^o \right\| \text{ subject to } k_n \geq \max\{0, d_{j+1} / d_j k_{j+1}\} + \varepsilon \text{ or } \varepsilon < k_j < d_{j+1} / d_j k_{j+1} - \varepsilon \tag{5}$$

for $j = n, \ldots, 2$, and

$$\min \left\| k_1 - k_1^o \right\| \text{ subject to } k_1 \geq \max\{0, d_2 / z_1 k_2\} + \varepsilon \quad , \tag{6}$$

where $\varepsilon > 0$ is a fixed parameter. (4)-(6) can be solved by the MATLAB routine *lsqlin* based on the active set method [1, 4]. Therefore, if the conditions in Theorem 1 hold, then we can solve (4)-(6) for $k_n, \ldots k_1$ successively. Then we compute $m_n, \ldots m_1$ by (3). Otherwise, the given data $(i\omega, z)$ is not generic.

3. Solvability of Problem A2

Notice that the data $(i\alpha, x)$ and $(i\beta, y)$ satisfy

$$(K - \alpha^2 M)x = 0 \text{ and } (K - \beta^2 M)y = 0 \ . \tag{7}$$

By (1), we can reformulate (7) as the following *2n* equations:

$$\begin{cases}\alpha^2 x_n m_n = \phi_n k_n \\ \beta^2 y_n m_n = \varphi_n k_n\end{cases} \text{ and } \begin{cases}\alpha^2 x_j m_j - \phi_j k_j = -\phi_{j+1} k_{j+1} \\ \beta^2 y_j m_j - \varphi_j k_j = -\varphi_{j+1} k_{j+1}\end{cases} \text{ for } j = n-1, \ldots, 1, \qquad (8)$$

where $\phi_j := x_j - x_{j-1}$ and $\varphi_j := y_j - y_{j-1}$ for $j = n-1, \ldots, 1$ with $x_0 = y_0 = 0$. Define the quantities

$$\Psi_j := \begin{bmatrix} \alpha^2 x_j & -\phi_j \\ \beta^2 y_j & -\varphi_j \end{bmatrix} \text{ and } \begin{pmatrix} a_j \\ b_j \end{pmatrix} := \Psi_j^{-1} \begin{pmatrix} \phi_{j+1} \\ \varphi_{j+1} \end{pmatrix} \text{ for } j = n-1, \ldots, 1, \qquad (9)$$

Then, by (8)-(9), we can get the following result.

Theorem 2. Suppose that $0 \neq \alpha, \beta \in \mathbb{R}, 0 \neq x, y \in \mathbb{R}^n$ and all matrices Ψ_j are nonsingular. Then Problem A2 is solvable if and only if the data $(i\alpha, x)$ and $(i\beta, y)$ satisfies the following conditions successively.

1) $x_n \phi_n > 0, y_n \varphi_n > 0$ and $\alpha^2 x_n \varphi_n = \beta^2 y_n \phi_n$. In this case, $k_n > 0$ is arbitrary and $m_n = \phi_n / (\alpha^2 x_n) k_n$.

2) For $j = n-1, \ldots, 1$, $a_j < 0$ and $b_j < 0$. In this case, $m_j = -a_j k_{j+1}$ and $k_j = -b_j k_{j+1}$ for some $k_{j+1} > 0$.

We remark that, under the conditions in Theorem 2, the physical mass and stiffness parameters can be determined uniquely if the total mass $w := \sum_{j=1}^n m_j$ is given. Given the total mass $w := \sum_{j=1}^n m_j$. Let

$$\tilde{k}_j := k_j / m_n \text{ and } \tilde{m}_j := m_j / m_n \text{ for } j = n, \ldots, 1. \qquad (10)$$

Then it is easy to check from (8) that the parameters $\tilde{m}_j$ and $\tilde{k}_j$ with $j = n, \ldots, 1$ are determined by $\tilde{k}_n = \alpha^2 x_n / \phi_n$, for $j = n, \ldots, 1$, $\tilde{m}_j = -a_j \tilde{k}_{j+1}$ and $\tilde{k}_j = -b_j \tilde{k}_{j+1}$ for some $\tilde{k}_{j+1} > 0$. By (10), the unique positive parameters m_j and k_j are given by $k_j = w / \tilde{w} \tilde{k}_j$ and $m_j = w / \tilde{w} \tilde{m}_j$ for $j = n, \ldots, 1$, where $\tilde{w} := \sum_{j=1}^n \tilde{m}_j$.

Acknowledgments The author's research was partially supported by the National Natural Science Foundation of China grant 10601043 and Program for New Century Excellent Talents in Xiamen University.

References

[1] R. Bro, S. de Jong, "*A fast non-negativity-constrained least squares algorithm*", J. Chemometrics, 11 (1997), pp. 393-401.

[2] M.T. Chu, N. Del Buono, and B. Yu, "*Structured quadratic inverse eigenvalue problem, I. Serially linked systems*", SIAM J. Sci. Comput., 29 (2007), pp. 2668-2685.

[3] G.M.L. Gladwell, "*Inverse Problems in Vibration*", Kluwer Academic Publishers, 2004.

[4] C.L. Lawson and R.J. Hanson, "*Solving Least Squares Problems*", Prentice-Hall, 1974.

[5] P. Nylen, "*Inverse eigenvalue problem: existence of special mass-damper-spring systems*", Linear Algebra Appl., 297 (1999), pp. 107-132.