Symmetric Tridiagonal Inverse Quadratic Eigenvalue Problems With Partial Eigendata

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The 3rd EASIAM Conference, Nov. 2-5, 2007, Xiamen, China

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Introduction

In vibration and structural analysis, we often need to solve a linear second-order differential equation (e.g. Finite Element Model)

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = \mathbf{0},$$

where $M, C, K \in \mathbb{C}^{n\times n}$ and $x(t)$ is an nth-order vector. The separation of variables $\underline{x(t) = xe^{\lambda t}}$ gives rise to the quadratic eigenvalue problem (See Tisseur'01)

$$Q(\lambda)x \equiv (\lambda^2 M + \lambda C + K)x = \mathbf{0}.$$

Inverse Quadratic eigenvalue Problems (IQEP):

- Predicted frequencies & model shapes(eigenvalues/eigenvectors) often disagree with that of experimentally measured from a realizable practical structure

- Reconstructing the quadratic pencil

$$Q(\lambda) \equiv \lambda^2 M + \lambda C + K$$

from experimentally measured frequencies/mode shapes.

Tridiagonal IQEPs

- Construct a nontrivial quadratic pencil

$$Q(\lambda) = \lambda^2 I + \lambda C + K$$

from a set of measured eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$, where

$$C = \begin{bmatrix} a_1 & -b_2 & & & \\ -b_2 & a_2 & -b_3 & & \\ \cdots & \cdots & \cdots & \ldots & \ldots \\ & & & -b_n & a_n \end{bmatrix}, K = \begin{bmatrix} c_1 & -d_2 & & & \\ -d_2 & c_2 & -d_3 & & \\ \cdots & \cdots & \cdots & \ldots & \ldots \\ & & & -d_n & c_n \end{bmatrix}$$

Applications: Vibration Systems (Nylen99, Gladwell04)

Difficulty: In practice, the physically realizable parameters $\{a_i\}_1^n$, $\{b_i\}_2^n$, $\{c_i\}_1^n$, and $\{d_i\}_2^n$ should be positive such that the corresponding C and K should be weakly diagonally dominant.

• Ram and Elhay (1996) determined the parameters from two sets of eigenvalues, where the positiveness is not necessarily preserved.

Problem Reformulation

Given the eigendata $(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ with

$$\Lambda = \mathrm{diag}\{\lambda_1^{[2]}, \ldots, \lambda_s^{[2]}, \lambda_{s+1}, \ldots, \lambda_p\}, \lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2},\ 1 \le i \le s,$$

$$X = [x_{1R}, x_{1I}, \ldots, x_{sR}, x_{sI}, x_{s+1}, \ldots, x_p]$$

The IQEP becomes

$$\begin{array}{ll} \min & \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} & X\Lambda^2 + CX\Lambda + KX = 0, \\ & C, K \in \Omega. \end{array} \tag{1}$$

Ω:The set of tridiagonal and weakly diagonally dominant matrices with the positive diagonals and negative off-diagonals.

Attn: The eigendata (Λ, X) are experimentally measured which often corrupted by noise.

To reduce the sensitivity, we solve the quadratically constrained quadratic problem (QCQP)

$$\begin{array}{ll} \text{min} & \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} & \|X\Lambda^2 + CX\Lambda + KX\| \leq \delta_n, \\ & C, K \in \Omega. \end{array} \tag{2}$$

where δ_n is a positive parameter depending on the noise level of the measured eigendata. Attn: $\delta_n \to 0$, problem (1) is recovered.

Since $C, K \in \Omega$, Problem (2) is reduced to

$$\begin{aligned} \min \quad & f_0(y) := \frac{1}{2}\|y - y^o\|^2 \\ \text{s.t.} \quad & f_1(y) := \|Ay - g\|^2 - \delta_n^2 \leq 0, \\ & f_2(y) := By \geq 0, \\ & y \geq 0, \end{aligned} \tag{3}$$

where the matrix A and the vector g are given in terms of the eigendata, $By \geq 0$ corresponds to the weakly diagonally dominant constraint, where

$y_o = (y_o^1, y_o^2, \ldots, y_o^n)^T \in \mathbb{R}^{4n-2}$ with $y_o^1 = (a_1^o, c_1^o)^T$ and $y_0^i = (a_i^o, b_i^o, c_i^o, d_i^o)^T$ for $2 \leq i \leq n$,
$y = (y^1, y^2, \ldots, y^n)^T \in \mathbb{R}^{4n-2}$ with $y^1 = (a_1, c_1)^T$ and $y^i = (a_i, b_i, c_i, d_i)^T$ for $2 \leq i \leq n$.

• Problem (3) is equivalent to finding $y \geq 0$, $\xi \geq 0$, and $\zeta \geq 0$ such that

$$\text{KKT:} \begin{cases} \nabla f_0(y) + \xi \nabla f_1(y) - \nabla f_2(y)^T \zeta = 0, \\ \xi \geq 0, \quad \zeta \geq 0, \quad -f_1(y) \geq 0, \quad f_2(y) \geq 0, \\ \xi f_1(y) = 0, \quad f_2(y)^T \zeta = 0. \end{cases} \tag{4}$$

Let

$$\mathcal{K} := \mathbb{R}_+^{4n-2} \times \mathbb{R}_+ \times \mathbb{R}_+^{2n}, \quad m := 6n - 1$$

$$F(z) := \begin{pmatrix} \nabla f_0(y) + \xi \nabla f(y) - \nabla f_2(y)^T \zeta \\ -f_1(y) \\ f_2(y) \end{pmatrix}, \quad z := (y, \xi, \zeta)$$

Solving (4) is to find a vector $z^* \in \mathcal{K}$ such that

$$\text{Variational inequalities: } (z - z^*)^T F(z^*) \geq 0, \text{ for all } z \in \mathcal{K} \quad (5)$$

or Robinson's normal equation:

$$F_0(z) := F(\Pi_{\mathcal{K}}(z)) + z - \Pi_{\mathcal{K}}(z) = 0. \quad (6)$$

in the sense that if $\hat{z}^*$ is a solution of (6), then $\underline{z^* := \Pi_{\mathcal{K}}(\hat{z}^*)}$ is a solution of (5), and conversely if z^* is a solution of (5), then $\hat{z}^* := z^* - F(z^*)$ is a solution of (6), see (Robinson92).

Smoothing Approx: By using the Chen-Harker-Kanzow-Smale smoothing function (Chen & Harker93) for $\Pi_{\mathcal{K}}(\cdot)$, we get the smoothing approximation for $F_0(\cdot)$:

$$\widetilde{G}(\epsilon, z) := F(p(\epsilon, z)) + z - p(\epsilon, z), \quad (\epsilon, z) \in \mathbb{R} \times \mathbb{R}^m \tag{7}$$

where

$$p(\epsilon, z) = \text{vec}\{\varphi(\epsilon, z_i)\},$$

$$\varphi(\epsilon, x) := \frac{1}{2}\left(x + \sqrt{x^2 + 4\epsilon^2}\right) \to x_+ \text{ as } \epsilon \to 0, \quad (\epsilon, x) \in \mathbb{R} \times \mathbb{R}.$$

To prevent the singularity, we define the regularized function $H : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+1}$ by

$$H(w) := \begin{pmatrix} \epsilon \\ G(w) \end{pmatrix}, \quad w := (\epsilon, z) \in \mathbb{R} \times \mathbb{R}^m, \tag{8}$$
$$G(w) := \widetilde{G}(w) + \epsilon z.$$

$z^* = (y^*, \xi^*, \zeta^*)$ solves (6) $\iff$ $w^* = (0, z^*)$ solves

$$H(w) = 0$$

Regularized Smoothing Newton Method

- $\bar{\epsilon} \in \mathbb{R}_{++}$ and $\tau \in (0, 1)$ such that $\tau\bar{\epsilon} < 1$.
- $\bar{w} := (\bar{\epsilon}, 0, 0) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}$.
- $\phi(w) := \|H(w)\|^2$ and $\psi(w) := \tau \min(1, \phi(w))$.

$$\begin{array}{c} H(w^{(k)}) + H'(w^{(k)})\Delta w^{(k)} = \psi(w^{(k)})\bar{w} \\ \phi(w^{(k)} + \delta^l \Delta w^{(k)}) \leq [1 - 2\sigma(1 - \tau\bar{\epsilon})\delta^l]\phi(w^{(k)}) \end{array} \tag{9}$$

Our Work

- **Theorem:** $\forall w = (\epsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$, $H'(w)$ is nonsingular.

- **Global Convergence:** Suppose the solution set of (6) is nonempty. Then the infinite sequence $\{w^{(k)}\}$ generated by our algorithm is bounded and any accumulation point w^* of $\{w^{(k)}\}$ is a solution of $H(w) = 0$.

- **Quadratic Convergence:** Suppose that w^* is an accumulation point of the sequence $\{w^{(k)}\}$ generated by our algorithm. If all $V \in \partial H(w^*)$ are nonsingular, then the whole sequence $\{w^{(k)}\}$ converges to w^* with

$$\|w^{(k+1)} - w^*\| = O(\|w^k - w^*\|^2), \quad \epsilon^{(k+1)} = O\Big((\epsilon^{(k)})^2\Big).$$

Numerical Results in Engineering Applications

- A damped mass-spring system governed by (see also Ram and Elhay'96)

$$I\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0,$$

$$C = P\text{diag}(0, e_1, e_2, \ldots, e_{n-1})P^T + \text{diag}(\pi_1, \ldots, \pi_n)$$
$$K = P\text{diag}(0, f_1, f_2, \ldots, f_{n-1})P^T + \text{diag}(\kappa_1, \ldots, \kappa_n)$$

where $P = [\delta_{ij} - \delta_{i+1,j}]$.

- $\{e_i^o\}_1^{n-1}$, $\{\pi_i^o\}_1^n$, $\{f_i^o\}_1^{n-1}$, $\{\kappa_i^o\}_1^n$, and $\{(\lambda_i, x_i)\}_1^p$ generated randomly.

• An upper bound estimate for the noise parameter δ_n (See Abdalla, Grigoriadis, and Zimmerman'00):

$$\delta_n = r(\|X\Lambda^2\| + \|C_o X\Lambda\| + \|K_o X\|), \quad r = 0.08$$

•

$$a) : \epsilon^{(0)} = \bar{\epsilon}, \quad z^{(0)} = \mathbf{0}; \quad b) : \epsilon^{(0)} = \bar{\epsilon}, \quad z^{(0)} = \mathbf{1}, \quad \bar{\epsilon} = 0.1.$$

•

$$\delta = 0.5, \quad \sigma = 0.5 \times 10^{-4}, \quad \tau = 0.2 \times \min(1, 1/\bar{\epsilon}).$$

Stopping criterion: $\underline{\|H(w^{(k)})\| \le 10^{-6}}$

$p = 15,\ s = 3$				
SP.	n	IT.	NF.	VAL.
a)	50	10	16	7.8×10^{-13}
	100	9	10	2.1×10^{-12}
	200	10	16	2.0×10^{-11}
	300	10	15	5.3×10^{-10}
	400	8	9	1.4×10^{-9}
	500	9	11	4.0×10^{-9}
b)	50	11	17	1.6×10^{-9}
	100	12	21	3.7×10^{-12}
	200	12	21	1.2×10^{-10}
	300	11	18	5.2×10^{-7}
	400	11	18	3.0×10^{-9}
	500	11	17	6.2×10^{-7}

$n = 100$					
SP.	p	s	IT.	NF.	VAL.
a)	10	3	8	9	5.3×10^{-12}
	20	6	9	11	3.5×10^{-12}
	30	9	9	11	1.2×10^{-11}
	40	12	9	12	6.6×10^{-12}
	50	15	9	12	7.1×10^{-12}
b)	10	3	9	10	4.9×10^{-12}
	20	6	11	19	1.5×10^{-8}
	30	9	10	13	1.2×10^{-11}
	40	12	12	22	7.0×10^{-12}
	50	15	12	21	7.3×10^{-12}

Conclusions

- A tridiagonal inverse quadratic eigenvalue problem is discussed.
- The problem is reformulated as a variational inequality.
- A regularized smoothing Newton method is proposed.
- Numerical experiments show the efficiency of our approach.

Thank You!