Nonnegative Inverse Eigenvalue Problems with Partial Eigendata

Zheng-Jian Bai

School of Mathematical Sciences

Xiamen University, Xiamen, China

zjbai@xmu.edu.cn

*Joint work with Stefano Serra-Capizzano (Università dell'Insubria, Italy) and Zhi Zhao (Xiamen University)

Zhengzhou, September 19-22, 2011.

Motivation

• Given a self-conjugate set of complex numbers $\{\lambda_i^*\}_{i=1}^n$ or partial eigenpairs $\{(\lambda_i^*, \boldsymbol{x}_i^*)\}_{i=1}^p$, the inverse eigenvalue problem (IEP) is to find a matrix $A \in \mathbb{R}^{n\times n}$ such that

$$\sigma(A) = \{\lambda_i^*\}_{i=1}^n \quad \text{or} \quad A\boldsymbol{x}_i^* = \lambda_i^* \boldsymbol{x}_i^*,\ i = 1, \ldots, p\ (p \leq n).$$

• IEPs arise in a wide variety of applications such as structural dynamics, control design, system identification, seismic tomography, remote sensing, geophysics, particle physics, and circuit theory, etc.

Attn:

• In practice, only partial eigendata can be obtained.

• In many applications, the matrix should preserve specific structures: symmetry, definiteness, sparsity or bandedness, nonnegativity, etc.

- Nonnegative matrices play an important role in many applications such as game theory, Markov chain, probabilistic algorithms, numerical analysis, discrete distributions, categorical data, group theory, matrix scaling, and economics, etc.

- Attn:
  - The nonnegative inverse eigenvalue problem (NIEP) has got much attention since 1940s.

Most of previous work are concerned with the following NIEP:

P.: Given a self-conjugate set of complex numbers $\{\lambda_j^*\}_{j=1}^n$, find a nonnegative matrix $A \in \mathbb{R}^{n\times n}$ such that $\sigma(A) = \{\lambda_j^*\}_{j=1}^n$.

• Attn: There exist few numerical algorithms. To our knowledge, only the isospectral flow method and the alternating projection method.

• For the applications, solvability and numerical methods of the NIEP, we have the following references.

- M. T. Chu and G. H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications*, Oxford University Press, Oxford, 2005.
- P. D. Egleston, T. D. Lenker, and S. K. Narayan, *The nonnegative inverse eigenvalue problem*, Linear Algebra Appl. 379 (2004), pp. 475–490.
- X. Chen and D. L. Liu, *Isospectral flow method for nonnegative inverse eigenvalue problem with prescribed structure*, J. Comput. Appl. Math., 235 (2011), pp. 3990–4002.
- M. T. Chu, F. Diele, and I. Sgura, *Gradient flow method for matrix completion with prescribed eigenvalues*, Linear Algebra Appl., 379 (2004), pp. 85–112.
- M. T. Chu and K. R. Driessel, *Constructing symmetric nonnegative matrices with prescribed eigenvalues by differential equations*, SIAM J. Math. Anal., 22 (1991), pp. 1372–1387.
- M. T. Chu and Q. Guo, *A numerical method for the inverse stochastic spectrum problem*, SIAM J. Matrix Anal. Appl., 19 (1998), pp. 1027–1039.
- R. Orsi, *Numerical methods for solving inverse eigenvalue problems for nonnegative matrices*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 190–212.

Nonnegative Inverse Problem with Partial Eigendata

- In this talk, we consider the nonnegative inverse problem with partial eigendata:

NIEP: Given a set of measured eigendata $\{(\lambda_j, x_j)\}_{j=1}^{p}$ ($p \ll n$). Find $A \in \mathbb{R}_{+}^{n\times n}$ such that

$$Ax_j = \lambda_j x_j \quad j = 1, \ldots, p.$$

- The eigendata $\{(\lambda_j, x_j)\}_{j=1}^{p}$ can be written as real form

$$(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}.$$

- Then the NIEP is to find $A \in \mathbb{R}_{+}^{n\times n}$ such that

$$AX = X\Lambda.$$

Nonsmooth Newton-type Method

• Let $K = X^T$ and $B = (X\Lambda)^T$. Then $\overline{A} \in \mathbb{R}^{n\times n}$ is a solution to the NIEP iff $\overline{Y} = \overline{A}^T$ is a global solution of the following optimization problem

$$\begin{array}{ll} \min & f(Y) := \frac{1}{2}\|KY - B\|_F^2 \\ \text{s.t.} & Y \in \mathbb{R}_+^{n\times n} \end{array}$$

with $f(\overline{Y}) = 0$.

• The KKT condition:

$$\mathcal{J}_Y f(Y) - Z = 0, \quad Y \in \mathbb{R}_+^{n\times n}, \quad Z \in \mathbb{R}_+^{n\times n}, \quad \langle Y, Z\rangle = 0.$$

• Let $F(Y) := \mathcal{J}_Y f(Y) = K^T(KY - B)$. Then the KKT condition becomes the complementarity problem:

$$Y \in \mathbb{R}_+^{n\times n}, \quad F(Y) \in \mathbb{R}_+^{n\times n}, \quad \langle Y, F(Y)\rangle = 0.$$

• Yamashita and Fukushima'97 and De Luca, Facchinei, and Kanzow'96 proposed generalized Newton methods for solving the following nonlinear complementarity problem: Find $x \in \mathbb{R}^n$ such that

$$\mathbb{R}^n \ni x \geq 0, \quad \mathbb{R}^n \ni F(x) \geq 0, \quad x^T F(x) = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function.

• By following their idea, we propose a nonsmooth Newton method for the NIEP.

• By using the well-known Fischer function

$$w(a,b) := \sqrt{a^2+b^2} - (a+b) \quad \text{for } a,b \in \mathbb{R},$$

which has the nice property:

$$w(a,b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0,\ ab = 0,$$

the KKT condition becomes the nonsmooth equation

$$\Phi(Y) = 0, \quad \Phi_{ij}(Y) := w(Y_{ij}, F_{ij}(Y)) \text{ for } i,j = 1,\ldots,n.$$

• Also, define the merit function $\theta : \mathbb{R}^{n\times n} \to \mathbb{R}$ by

$$\theta(Y) := \frac{1}{2}\|\Phi(Y)\|_F^2.$$

• We now recall the definition of Clarke's generalized Jacobian [Clarke'83].

• Let $\mathcal{Y}$ and $\mathcal{Z}$ be two finite dimensional real vector spaces, each equipped with a scalar inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $\mathcal{D}$ be an open set in $\mathcal{Y}$ and $G : \mathcal{D} \to \mathcal{Z}$ be a locally Lipschitz continuous function on $\mathcal{D}$. Rademacher's theorem says that G is Fréchet differentiable almost everywhere in $\mathcal{D}$.

• Clarke's generalized Jacobian of G at $y \in \mathcal{D}$ is given by

$$\partial G(y) := \mathrm{conv}\{\partial_B G(y)\},$$

where "conv" means the convex hull and

$$\partial_B G(y) = \{\lim_{j \to \infty} \mathcal{J}_y G(y^j) \ : \ y^j \to y, \ G \text{ is Fréchet differentiable at } y^j \in \mathcal{D}\}.$$

• On Clarke's generalized Jacobian of Φ, we have the following result.

• **Theorem (Bai, Serra, and Zhao'11)** Let $Y \in \mathbb{R}^{n\times n}$. Then, for any $H \in \mathbb{R}^{n\times n}$,

$$\partial\Phi(Y)H \subseteq \Gamma(Y)\circ H + \Omega(Y)\circ \mathcal{J}_Y F(H),$$

where $\Gamma(Y)$ and $\Omega(Y)$ are two n-by-n matrices with entries determined by

$$\Gamma_{ij}(Y) = \frac{Y_{ij}}{\|(Y_{ij}, F_{ij}(Y))\|} - 1, \quad \Omega_{ij}(Y) = \frac{F_{ij}(Y)}{\|(Y_{ij}, F_{ij}(Y))\|} - 1,$$

if $(Y_{ij}, F_{ij}(Y)) \neq (0,0)$ and by

$$\Gamma_{ij}(Y) = \Gamma_{ij} - 1, \quad \Omega_{ij}(Y) = \Omega_{ij} - 1 \quad \text{for any } (\Gamma_{ij}, \Omega_{ij}) \text{ s.t. } \|(\Gamma_{ij}, \Omega_{ij})\| \leq 1$$

if $Y_{ij} = 0 = F_{ij}(Y)$.

• We also need the equivalent definition of semismoothness [Qi and Sun'93].

• Let $G : \mathcal{D} \subseteq \mathcal{Y} \to \mathcal{Z}$ be a locally Lipschitz continuous function on the open set $\mathcal{D}$.

1) G is said to be semismooth at $y \in \mathcal{D}$ if G is directionally differentiable at y and for any $V \in \partial G(y+h)$ and $h \to 0$,

$$G(y+h) - G(y) - V(h) = o(\|h\|).$$

2) G is said to be strongly semismooth at $y \in \mathcal{D}$ if G is semismooth at y and for any $V \in \partial G(y+h)$ and $h \to 0$,

$$G(y+h) - G(y) - V(h) = O(\|h\|^2).$$

• **Theorem (Bai, Serra, and Zhao'11)** Φ is strongly semismooth.

• **Theorem (Bai, Serra, and Zhao'11)** θ is continuously differentiable and its gradient at $Y \in \mathbb{R}^{n\times n}$ is given by

$$\begin{aligned} \nabla\theta(Y) &= \{V^*\Phi(Y) \mid V \in \partial\Phi(Y)\} \\ &\subseteq \Gamma(Y)\circ\Phi(Y) + (\mathcal{J}_Y F)^*(\Omega(Y)\circ\Phi(Y)). \end{aligned}$$

• **Theorem (Bai, Serra, and Zhao'11)** The transpose of any stationary point of θ is a solution to the NIEP.

Observation:

• Let $Y \in \mathbb{R}^{n\times n}$. Then, every element in

$$\mathcal{H}(Y) := \{D_1 \circ + D_2 \circ \mathcal{J}_Y F \mid -D_1, -D_2 \in \mathbb{R}^{n\times n}_{++}\}$$

is nonsingular.

• We now construct a subset of $\partial_B \Phi(Y)$, which is easy to evaluate. Define

$$\mathcal{L}(Y) := \{L = S \circ + T \circ \mathcal{J}_Y F \mid (S, T) \in \mathcal{G}(Y)\},$$

$\mathcal{G}(Y) := \{(S, T) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \mid (S, T) = (S(Y, Z), T(Y, Z)), Z \in \mathcal{Z}(Y)\},$

where $\mathcal{Z}(Y) := \{Z \in \mathbb{R}^{n\times n} \mid Z_{ij} \neq 0 \text{ if } (Y_{ij}, F_{ij}(Y)) = (0, 0)\}$ and

$$
\begin{aligned}
&(S_{ij}(Y,Z), T_{ij}(Y,Z)) \\
:= &\begin{cases}
\left(\frac{Z_{ij}}{\sqrt{Z_{ij}^2 + (\mathcal{J}_Y F(Z))_{ij}^2}} - 1, \frac{(\mathcal{J}_Y F(Z))_{ij}}{\sqrt{Z_{ij}^2 + (\mathcal{J}_Y F(Z))_{ij}^2}} - 1 \right), & \text{if } Y_{ij} = 0 = F_{ij}(Y), \\
\left(\frac{Y_{ij}}{\sqrt{Y_{ij}^2 + F_{ij}^2(Y)}} - 1, \frac{F_{ij}(Y)}{\sqrt{Y_{ij}^2 + F_{ij}^2(Y)}} - 1 \right), & \text{otherwise.}
\end{cases}
\end{aligned}
$$

- **Theorem (Bai, Serra, and Zhao'11)** For any $H \in \mathbb{R}^{n\times n}$,

$$\mathcal{L}(Y)H \subseteq \partial_B \Phi(Y)H.$$

- Attn: $S_{ij}, T_{ij} \in [-2, 0]$. Any $L \in \mathcal{L}(Y)$ is not necessarily nonsingular.

- Define

$$\widetilde{\mathcal{L}}(Y) := \{\widetilde{L} = (S + \Delta S) \circ +(T + \Delta T) \circ \mathcal{J}_Y F \mid (S, T) \in \mathcal{G}(Y), ((\Delta S)_{ij}, (\Delta T)_{ij}) \in \mathcal{Q}(Y, S_{ij}, T_{ij}) \text{ for } i, j = 1, \ldots, n\},$$

where

$$\mathcal{Q}(Y, S_{ij}, T_{ij}) := \begin{cases} \left\{((\Delta S)_{ij} = \frac{\zeta(\theta(Y))}{T_{ij}}, (\Delta T)_{ij} = 0)\right\}, \text{if } -\delta < S_{ij} \text{ and } T_{ij} \le -\delta, \\ \left\{((\Delta S)_{ij} = \mu \frac{\zeta(\theta(Y))}{T_{ij}}, (\Delta T)_{ij} = (1-\mu)\frac{\zeta(\theta(Y))}{S_{ij}}), \mu \in [0,1]\right\}, \\ \qquad \text{if } S_{ij} \le -\delta \text{ and } T_{ij} \le -\delta, \\ \left\{((\Delta S)_{ij} = 0, (\Delta T)_{ij} = \frac{\zeta(\theta(Y))}{S_{ij}})\right\}, \text{if } S_{ij} \le -\delta \text{ and } -\delta < T_{ij}. \end{cases}$$

• $\delta > 0$ is a threshold.

• $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function such that $\zeta(0) = 0$ and $\zeta(t) > 0$ for all $t > 0$.

• If $\theta(Y) > 0$, then it is easy to know that

$$-(S + \Delta S), -(T + \Delta T) \in \mathbb{R}^{n \times n}_{++}.$$

In this case, any element in $\widetilde{\mathcal{L}}(Y)$ is nonsingular.

Nonsmooth Newton-type Method

Step 0. Give $Y^0 \in \mathbb{R}^{n\times n}$, $\eta \in (0,1)$, and $\rho, \sigma \in (0, 1/2)$. $k := 0$.

Step 1. If $\theta(Y^k) = 0$, then stop. Otherwise, go to Step 2.

Step 2. Select an element $\widetilde{L}_k \in \widetilde{\mathcal{L}}(Y^k)$. Apply an iterative method (e.g., the transpose-free quasi-minimal residual (TFQMR) method) to solve

$$\widetilde{L}_k(\Delta Y^k) + \Phi(Y^k) = 0 \leftarrow \text{Inexact Newton Step} \tag{1}$$

for $\Delta Y^k \in \mathbb{R}^{n\times n}$ such that

$$\|\widetilde{L}_k(\Delta Y^k) + \Phi(Y^k)\|_F \le \eta_k \|\Phi(Y^k)\|_F, \tag{2}$$

and

$$\langle \nabla\theta(Y^k), \Delta Y^k\rangle \le -\eta_k \langle \Delta Y^k, \Delta Y^k\rangle, \tag{3}$$

where $\eta_k := \min\{\eta, \|\Phi(Y^k)\|_F\}$. If (2) and (3) are not attainable, then let

$$\Delta Y^k := -\nabla\theta(Y^k).$$

Step 3. Let l_k be the smallest nonnegative integer l such that

$$\theta(Y^k + \rho^l \Delta Y^k) - \theta(Y^k) \leq \sigma \rho^l \langle \nabla\theta(Y^k), \Delta Y^k \rangle. \leftarrow \text{Line Search Step}$$

Set

$$Y^{k+1} := Y^k + \rho^{l_k} \Delta Y^k.$$

Step 4. Replace k by $k+1$ and go to Step 1.

• **Theorem (Bai, Serra, and Zhao'11)** Global Convergence: The transpose of any accumulation point of the sequence $\{Y^k\}$ generated by our Algorithm is a solution to the NIEP.

• **Theorem (Bai, Serra, and Zhao'11)** Quadratic Convergence: Let $\overline{Y}$ be an accumulation point of the sequence $\{Y^k\}$ generated by our Algorithm. Suppose that all elements in $\partial_B \Phi(\overline{Y})$ are nonsingular, and the function ζ is such that $\zeta(t) = O(\sqrt{t})$. Then the sequence $\{Y^k\}$ converges to $\overline{Y}$ quadratically.

• Nonsingularity Conditions of $\partial\Phi(\cdot)$: Let

$$\begin{cases} \alpha := \{(i,j) \mid \overline{Y}_{ij} > 0, F_{ij}(\overline{Y}) = 0\}, \\ \beta := \{(i,j) \mid \overline{Y}_{ij} = 0, F_{ij}(\overline{Y}) = 0\}, \\ \gamma := \{(i,j) \mid \overline{Y}_{ij} = 0, F_{ij}(\overline{Y}) > 0\}, \end{cases}$$

• **Theorem (Bai, Serra, and Zhao'11)** Let $\overline{Y}$ be an accumulation point of the sequence $\{Y^k\}$ generated by our Algorithm. Suppose that the strong second order sufficient condition holds at $\overline{Y}$ i.e.,

$$\langle d, \mathcal{J}^2_{YY} f(\overline{Y}) d\rangle > 0 \quad \forall\, d \in \{d \in \mathbb{R}^{n\times n} \mid d_\gamma = 0\}\backslash\{0\}.$$

Then any element in $\partial\Phi(\overline{Y})$ is nonsingular.

Remark:

- Suppose that X is full column rank.
- Let the QR decomposition of X be given by

$$X = U \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $U = [U_1, U_2]$ is an $n \times n$ orthogonal matrix with $U_1 \in \mathbb{R}^{n \times p}$ and R is a $p \times p$ nonsingular upper triangular matrix.

- If

$$U_1^T d \neq 0 \quad \forall\, d \in \{d \in \mathbb{R}^{n \times n} \,|\, d_\gamma = 0\} \backslash \{0\},$$

then the strong second order sufficient condition holds at $\overline{Y}$.

- In this case, any element in $\partial \Phi(\overline{Y})$ is nonsingular.

Extension 1: The Symmetric NIEP

- **SNIEP.** Construct a nontrivial n-by-n symmetric nonnegative matrix A from a set of measured eigendata $\{(\lambda_k, \boldsymbol{x}_k)\}_{k=1}^{p}$ ($p \le n$).

- Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^{p\times p}$ and $X = [x_1, \ldots, x_p] \in \mathbb{R}^{n\times p}$. Then the SNIEP is to find an $A \in \mathbb{SR}_+^{n\times n}$ such that

$$AX = X\Lambda.$$

- Let $K = X^T$ and $B = (X\Lambda)^T$. $\overline{A} \in \mathbb{SR}_+^{n\times n}$ is a solution to the SNIEP iff $\overline{Y} = \overline{A} \in \mathbb{SR}^{n\times n}$ is a global solution of the following optimization problem

$$\begin{cases} \min & f(Y) := \frac{1}{2}\|KY - B\|_F^2 \\ \text{s.t.} & Y \in \mathbb{SR}_+^{n\times n} \end{cases}$$

with $f(\overline{Y}) = 0$.

- The KKT condition:

$$\mathcal{J}_Y f(Y) - Z = 0, \quad Y \in \mathbb{SR}_+^{n\times n}, \quad Z \in \mathbb{SR}_+^{n\times n}, \quad \langle Y, Z\rangle = 0,$$

- Let $F(Y) := \mathcal{J}_Y f(Y) = \frac{1}{2}\big(K^T(KY - B) + (KY - B)^T K\big)$. The KKT condition is reduced to the following MCP

$$Y \in \mathbb{SR}_+^{n\times n}, \quad F(Y) \in \mathbb{SR}_+^{n\times n}, \quad \langle Y, F(Y)\rangle = 0.$$

- By using the well-known Fischer function, the MCP becomes

$$\Phi(Y) = 0, \quad \Phi_{ij}(Y) := w(Y_{ij}, F_{ij}(Y)) \quad \text{for } i, j = 1, \ldots, n.$$

- Also, define the merit function $\theta : \mathbb{SR}^{n\times n} \to \mathbb{R}$ by

$$\theta(Y) := \frac{1}{2}\|\Phi(Y)\|_F^2.$$

- Therefore, we can use our proposed method for the SNIEP.

Extension 2: The Case of Prescribed Entries

- The NIEP with prescribed entries is to find $A \in \mathbb{R}_+^{n\times n}$ such that

$$AX = X\Lambda, \quad A_{ij} = (A_a)_{ij} \quad \forall (i,j) \in \mathcal{I}, \tag{4}$$

where $A_a \in \mathbb{R}_+^{n\times n}$ is a prescribed matrix.

- For a matrix $M \in \mathbb{R}^{n\times n}$, let $M_{\mathcal{I}}$ be the column vector of the elements M_{ij} for all $(i,j) \in \mathcal{I}$ and define the linear operator $P : \mathbb{R}^{|\mathcal{I}|} \to \mathbb{R}^{n\times n}$ by

$$P_{ij}(M_{\mathcal{I}}) := \begin{cases} M_{ij}, & \text{if } (i,j) \in \mathcal{I} \\ 0, & \text{otheriwse.} \end{cases}$$

- Let $\mathcal{N} := \{(i,j) | i,j = 1,\ldots,n\}$ and $\mathcal{J} := \mathcal{N} \backslash \mathcal{I}$. Then, the NIEP with prescribed entries turns into a new NIEP, which is to find $A_{\mathcal{J}} \in \mathbb{R}_{+}^{|\mathcal{J}|}$ such that

$$P(A_{\mathcal{J}})X = X\Lambda - P((A_a)_{\mathcal{I}})X.$$

- Let $K := X^T$ and $B := (X\Lambda - P((A_a)_{\mathcal{J}})X)^T$. Define the index sets $\tilde{\mathcal{I}} \subseteq \mathcal{N}$ and $\tilde{\mathcal{J}} = \mathcal{N} \backslash \tilde{\mathcal{I}}$ such that

$$P(M_{\tilde{\mathcal{I}}}) = P(M_{\mathcal{I}})^T \quad \text{and} \quad P(M_{\tilde{\mathcal{J}}}) = P(M_{\mathcal{J}})^T \quad \forall\, M \in \mathbb{R}^{n\times n}.$$

• The NIEP with prescribed entries is to find a global solution to the following optimization problem

$$\begin{cases} \min & f(Y_{\tilde{\mathcal{J}}}) := \frac{1}{2}\|KP(Y_{\tilde{\mathcal{J}}}) - B\|_F^2 \\ \text{s.t.} & P(Y_{\tilde{\mathcal{J}}}) \in \mathbb{R}_+^{n\times n} \end{cases}$$

in the sense that $\overline{A}_{\mathcal{J}}$ is a solution to the NIEP with prescribed entries iff $\overline{Y}_{\tilde{\mathcal{J}}}$ is a global solution to the above optimization problem with $P(\overline{Y}_{\tilde{\mathcal{J}}}) = P(\overline{A}_{\mathcal{J}})^T$ and $f(\overline{Y}_{\tilde{\mathcal{J}}}) = 0$.

• The KKT condition:

$$\mathcal{J}_{Y_{\tilde{\mathcal{J}}}} f(Y_{\tilde{\mathcal{J}}}) - Z_{\tilde{\mathcal{J}}} = 0, \quad Y_{\tilde{\mathcal{J}}} \geq 0, \quad Z_{\tilde{\mathcal{J}}} \geq 0, \quad \langle Y_{\tilde{\mathcal{J}}}, Z_{\tilde{\mathcal{J}}} \rangle = 0,$$

- Let

$$F(Y_{\tilde{\mathcal{J}}}) := \mathcal{J}_{Y_{\tilde{\mathcal{J}}}} f(Y_{\tilde{\mathcal{J}}}) = (K^T(KY - B))_{\tilde{\mathcal{J}}}.$$

- The KKT condition becomes the following MCP

$$Y_{\tilde{\mathcal{J}}} \geq 0, \quad F(Y_{\tilde{\mathcal{J}}}) \geq 0, \quad \langle Y_{\tilde{\mathcal{J}}}, F(Y_{\tilde{\mathcal{J}}}) \rangle = 0.$$

- By using Fischer function, the MCP is equivalent to the following system

$$P(\Phi(Y_{\mathcal{J}})) = 0, \quad \Phi_j(Y_{\tilde{\mathcal{J}}}) := w((Y_{\tilde{\mathcal{J}}})_j, F_j(Y_{\tilde{\mathcal{J}}})) \text{ for } j = 1, \ldots, |\tilde{\mathcal{J}}|.$$

- Also, define the merit function $\theta : \mathbb{R}^{|\tilde{\mathcal{J}}|} \to \mathbb{R}$ by

$$\theta(Y_{\tilde{\mathcal{J}}}) := \frac{1}{2} \| P(\Phi(Y_{\tilde{\mathcal{J}}})) \|_F^2.$$

Numerical Tests

- We choose a matrix Z as

$$Z_{ij} = \begin{cases} 1, & \text{if } Y_{ij}^k = 0 = F_{ij}(Y^k), \\ 0, & \text{otherwise}, \end{cases} \qquad \forall\, i, j = 1, \ldots, n$$

- Starting point: $Y^0 = 0$.

- The stopping criteria: $\theta(Y^k) \leq 10^{-20}$.

- Other parameters: $\delta = 0.05$, $\zeta(t) = 0.1 \min\{1, t\}$, $\eta = 10^{-5}$, $\rho = 0.5$, and $\sigma = 10^{-4}$.

• Ex. 1: The NIEP: Let $n = 6$ and we randomly generate an n-by-n nonnegative matrix $\widehat{A}$ as follows:

$$\widehat{A} = \begin{bmatrix} 0.8270 & 0.3112 & 0.8260 & 0.9632 & 0.5067 & 0.1420 \\ 0.5522 & 1.0324 & 0.8392 & 0.3307 & 0.7635 & 0.6059 \\ 1.0387 & 0.4184 & 0.9698 & 0.4000 & 1.0901 & 0.4353 \\ 0.3360 & 0.4230 & 0.7811 & 0.9965 & 0.8516 & 0.6115 \\ 0.1277 & 0.5167 & 0.6465 & 0.8481 & 0.7110 & 0.5592 \\ 0.2316 & 0.7494 & 1.0024 & 0.8008 & 0.8709 & 0.8055 \end{bmatrix}.$$

The eigenvalues of $\widehat{A}$ are given by $\lambda_1 = 3.9752$, $\lambda_{2,3} = 0.6941 \pm 0.2340\sqrt{-1}$, $\lambda_4 = -0.2290$, $\lambda_{5,6} = 0.1039 \pm 0.0572\sqrt{-1}$. We use the first three eigenvalues $\{\lambda_i\}_{i=1}^3$ and associated eigenvectors $\{x_i\}_{i=1}^3$ as the prescribed eigendata.

- By applying Algorithm to Ex. 1, we obtain the physical solution:

$$\overline{A} = \begin{bmatrix} 0.8876 & 0.1437 & 0.8302 & 0.7295 & 0.4711 & 0.4780 \\ 0.5587 & 1.0091 & 0.7787 & 0.4828 & 0.4563 & 0.7778 \\ 0.9444 & 0.4134 & 0.9486 & 0.7746 & 0.5343 & 0.6240 \\ 0.5405 & 0.3926 & 0.5768 & 0.9097 & 0.7712 & 0.8098 \\ 0.3488 & 0.4854 & 0.4339 & 0.7298 & 0.6675 & 0.7527 \\ 0.5193 & 0.7965 & 0.6736 & 0.8095 & 0.7427 & 0.9366 \end{bmatrix}$$

with the error $\|\overline{A}X - X\Lambda\|_F < 4.6 \times 10^{-16}$.

• Ex. 2: The SNIEP with prescribed entries, which comes from the inverse problem in vibrations. Let $\widehat{A}$ be a symmetric tridiagonal oscillatory matrix of order n with $n = 6$, i.e.,

$$\widehat{A} = \begin{bmatrix} 4.7270 & 0.8246 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0.8246 & 4.4522 & 1.1618 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1.1618 & 4.9387 & 1.1349 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1.1349 & 4.2360 & 1.1497 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.1497 & 4.0277 & 0.6471 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0.6471 & 4.1316 \end{bmatrix}.$$

The eigenvalues of $\widehat{A}$ are given by $\{2.4625, 3.2861, 4.0556, 4.7689, 5.4343, 6.5059\}$. We use the last three eigenvalues $\{\lambda_i\}_{i=4}^{6}$ and associated eigenvectors $\{x_i\}_{i=4}^{6}$ as the prescribed eigendata. The off-tridiagonal entries of $\widehat{A}$ are seen as prescribed entries.

- By applying Algorithm to Ex. 2, we obtain the physical solution:

$$\overline{A} = \begin{bmatrix} 4.7270 & 0.8246 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0.8246 & 4.4522 & 1.1618 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1.1618 & 4.9387 & 1.1349 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1.1349 & 4.2360 & 1.1497 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.1497 & 4.0277 & 0.6471 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0.6471 & 4.1316 \end{bmatrix}$$

with the error $\|\overline{A}X - X\Lambda\|_F < 3.7 \times 10^{-13}$.

Table: Convergence results for Ex.1–Ex.2

	Ex. 1	Ex. 2
k	$\theta(Y^k)$	$\theta(Y^k)$
0	3.2×10^{1}	1.8×10^{2}
1	1.5×10^{0}	8.8×10^{0}
2	8.6×10^{-2}	5.2×10^{-1}
3	2.9×10^{-4}	7.1×10^{-3}
4	2.5×10^{-9}	3.5×10^{-6}
5	1.9×10^{-19}	1.6×10^{-13}
6	9.9×10^{-32}	4.0×10^{-27}

• Ex. 3: The NIEP for different n and p. Let $\widehat{A}$ be a random $n \times n$ nonnegative matrix with each entry generated from the uniform distribution on the interval $[0, 10]$. We choose the first p eigenvalues of $\widehat{A}$ with largest absolute values and associated eigenvectors as the prescribed eigendata.

• Ex. 4: The SNIEP with prescribed entries for different n and p. Let the matrix $\widehat{A}$ be a random $n \times n$ symmetric nonnegative matrix with entries $\widehat{A}_{ij} \in [0, 1]$ for all $i, j = 1 \ldots, n$. Let the entries of $\widehat{A}$ with the values equal to or greater than 0.85 as the prescribed entries. We choose the first largest p eigenvalues of $\widehat{A}$ and associated eigenvectors as the prescribed eigendata.

Ex.3: $p = 20$						
n	cputime	IT.	NF.	$\theta(Y^0)$	$\theta(\overline{Y})$	$\|\overline{A}X - X\Lambda\|_F$
100	1 s	8	9	5.1×10^5	5.6×10^{-27}	1.1×10^{-13}
200	3 s	9	10	2.0×10^6	4.6×10^{-26}	3.1×10^{-13}
500	20 s	10	11	1.3×10^7	1.9×10^{-23}	8.6×10^{-12}
1,000	46 s	8	9	5.0×10^7	1.9×10^{-23}	8.0×10^{-12}
1,500	1 m 26 s	8	9	1.1×10^8	1.3×10^{-22}	1.8×10^{-11}
2,000	4 m 05 s	10	11	2.0×10^8	1.6×10^{-24}	1.8×10^{-12}

$n = 500$						
p	cputime	IT.	NF.	$\theta(Y^0)$	$\theta(\overline{Y})$	$\|\overline{A}X - X\Lambda\|_F$
10	12 s	8	9	1.3×10^7	1.6×10^{-25}	5.5×10^{-13}
20	29 s	11	12	1.3×10^7	1.6×10^{-25}	5.7×10^{-13}
50	32 s	13	18	1.3×10^7	1.8×10^{-24}	1.9×10^{-12}
100	2 m 08 s	16	17	1.3×10^7	5.6×10^{-25}	1.0×10^{-12}
150	3 m 58 s	20	21	1.3×10^7	3.2×10^{-22}	4.9×10^{-11}
200	37 m 19 s	39	64	1.4×10^7	6.1×10^{-24}	6.4×10^{-12}

Ex.4: $p = 20$						
n	cputime	IT.	NF.	$\theta(Y^0)$	$\theta(\overline{Y})$	$\|\bar{A}X - X\Lambda\|_F$
100	1 s	6	7	4.2×10^3	5.6×10^{-22}	6.6×10^{-11}
200	2 s	7	8	1.7×10^4	2.1×10^{-28}	3.2×10^{-14}
500	11 s	7	8	1.0×10^5	1.1×10^{-27}	7.3×10^{-14}
1,000	34 s	7	8	4.1×10^5	1.3×10^{-25}	7.6×10^{-13}
1,500	1 m 15 s	7	8	9.1×10^5	1.1×10^{-23}	7.2×10^{-12}
2,000	2 m 07 s	7	8	1.6×10^6	1.4×10^{-22}	2.5×10^{-11}

$n = 500$						
p	cputime	IT.	NF.	$\theta(Y^0)$	$\theta(\overline{Y})$	$\|\bar{A}X - X\Lambda\|_F$
10	10 s	7	8	1.0×10^5	9.4×10^{-28}	6.6×10^{-14}
20	11 s	7	8	1.0×10^5	1.1×10^{-27}	7.6×10^{-14}
50	14 s	7	8	1.0×10^5	2.0×10^{-27}	1.0×10^{-13}
80	18 s	7	8	1.1×10^5	5.4×10^{-26}	1.4×10^{-13}
100	21 s	7	8	1.1×10^5	3.4×10^{-27}	1.4×10^{-13}

Z. J. Bai, S. Serra-Capizzano, and Z. Zhao, *Nonnegative inverse eigenvalue problems with partial eigendata*, Numer. Math., accepted.

Thank You for Your Attention!