Burgers' equation with high Reynolds number

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Burgers' equation, involving very high Reynolds numbers, is numerically solved by using a new approach based on the distributed approximating functional for representing spatial derivatives of the velocity field. For moderately large Reynolds numbers, this simple approach can provide very high accuracy while using a small number of grid points. In the case where the Reynolds number $\geqslant 10^5$, the exact solution is not available and a discrepancy exists in the literature. Our results clarify the behavior of the solution under these conditions. © *1997 American Institute of Physics.* [S1070-6631(97)02406-9]

This Brief Communication reports a new method for solving Burgers' equation[1] and clarifies an on-going disagreement regarding the early time behavior of solutions of Burgers' equation in the case of large Reynolds number ($\mathrm{Re}=10^5$).

Burgers' model of turbulence[1] is a very important fluid dynamical model both for the conceptual understanding of a class of physical flows and for testing various numerical algorithms. A great deal of effort has been expended in the past few years to compute efficiently the numerical solutions of the Burgers' equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

for large values of the Reynolds number (Re). One of the major difficulties is due to inviscid boundary layers produced by the steepening effect of the nonlinear advection term in Burgers' equation. This difficulty is also encountered in an inviscid Navier–Stokes equation for a convection dominated flow, and in fact, Burgers' equation is one of the principle equations used to test new numerical methods. For given initial-boundary conditions of Eq. (1) such as $u(x,0)=\sin(\pi x)$, $u(0,t)=u(1,t)=0$, and a Reynolds number equal to 10^5, the solution develops a sharp shock wave front at the boundary $x=1$ after a certain time of propagation. This thin boundary layer normally requires an enormous number of evenly spaced grid points to describe the numerical solution in the boundary layer region.[2] A typical signature of an insufficient number of points and poor accuracy is the appearance of oscillations in the solution in the boundary layer region. In their treatment of Burgers' equation, Varoglu and Finn[3] presented an isoparametric space-time finite element approach, utilizing the hyperbolic differential equation associated with Burgers' equation. They employed 200 elements and 5 iterations in order to solve the equation. Another approach which has been used by Nguyen and Reynen[4] is the least squares weak-formulation of the finite element method. Caldwell *et al.*[5] further developed the finite element method to allow for different sized elements at each stage of the calculation based on feed-back from the previous step. More recently, Kakuda and Tosaka proposed a generalized boundary element approach.[6] For very large Reynolds numbers ($\mathrm{Re}\approx 10^5$), these authors found significantly different behavior for the early time solutions ($t=0.2$, 0.4) from those of Varoglu and Finn,[3] and of Nguyen and Reynen.[4] Bar-Yoseph *et al.* more recently have discussed a number of space-time spectral element methods for solving Burgers' equation.[7] Arina and Canuto[8] have approached Burgers' equation using a self-adaptive, domain decomposition method called the χ-formulation. Various finite difference schemes for Burgers' equation have been compared by Biringen and Saati.[9] Shizgal *et al.*[10] also have compared different spectral method approaches to Burgers' equation.

In an earlier work,[11] we applied the recently developed distributed approximating functional (DAF) method[12,13] to Burgers' equation for moderately large Reynolds numbers. We refer the reader to that work for more details about the DAF method and its application to Burgers' equation. It was found that the DAF approach provides one of the most accurate numerical solutions yet available, while requiring a small number of grid points and permitting the use of reasonably large time steps. However for large Re values, a significant increase in the number of grid points as well as a reduction in the time increment is required for the solution to be stable. A new scheme based on the properties of the DAF is proposed in this Brief Communication to handle the cases where the Reynolds number is large. The accuracy and reliability of the present method is verified using the exact solutions of Burgers' equation[14,15] which are available for *moderately* large Reynolds numbers.

We proceed by introducing a C_2 mapping of the coordinate x, f: $y=f(x)$ and its inverse, $x=f^{-1}(y)$ such that the arbitrarily large but finite gradient of the solutions in the boundary layer region will be effectively reduced in the y-representation. Accordingly, Burgers' equation in the y-representation reads

$$\frac{\partial u}{\partial t}=\left[\frac{1}{\mathrm{Re}}\frac{\partial^2 y}{\partial x^2}-u\frac{\partial y}{\partial x}\right]_{x=f^{-1}(y)}\frac{\partial u}{\partial y}+\frac{1}{\mathrm{Re}}\left[\frac{\partial y}{\partial x}\right]^2_{x=f^{-1}(y)}\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

With an appropriate choice of the function f, the number of

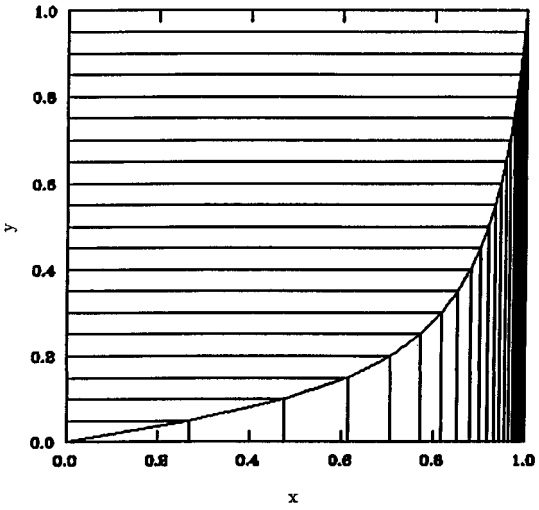

FIG. 1. The grid distributions in both x and y representations ($A = 1.45$).

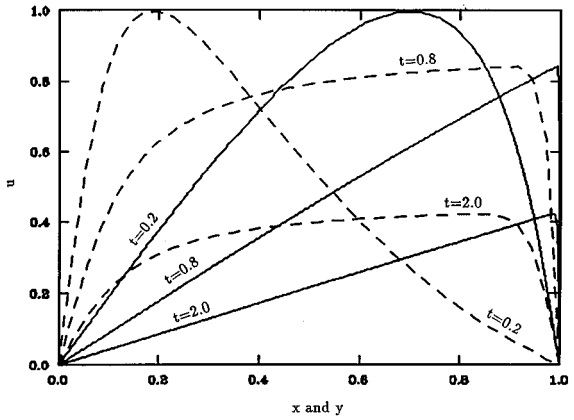

FIG. 2. The solutions of the Burger's equation for $\mathrm{Re}=10^3$ ($A=1.456$) using 35 grid points in either x (solid lines) or y (dash lines) representation.

grid points can be reduced by three orders of magnitude. It is to be noted that invertible mappings are widely used for distributing grid points to resolve the behavior accurately at boundaries.[16] This method becomes particularly powerful when combined with DAFs, since the DAF approach provides the accuracy of a spectral method on a uniform grid. For the aforementioned initial-boundary conditions we simply choose $y=\tan(Ax)/\tan(A)$, where A is a constant varying between 0 and $\pi/2$, depending on the size of the Reynolds number. For a small value of Reynolds number, such as $\mathrm{Re}=10$, we choose A close to 0, which distributes the grid points more or less evenly in both the x and y representations. For very large Reynolds numbers, such as $\mathrm{Re}=10^5$, we choose A close to $\pi/2$, which distributes grid points densely in the boundary layer region in the x representation for an evenly spaced y representation grid. The grid point distributions in both the x and y representations are plotted in Fig. 1 for $A=1.45$. In the present computation, we use a second order approximation[11] and take the Hermite DAF parameters as $M=88$ and $\sigma=3.05\Delta$ for all calculations.

Numerically, Burgers' equation can be solved most easily in the y representation according to Eq. (2), since $u(y)$ is a slowly varying function. This is illustrated in Fig. 2 for the moderately large Reynolds number of 10^3. It is noted that only 35 grid points are required to obtain the present solutions, which are oscillation-free. This typically requires 100 elements (or grid points) using other methods. As seen in Fig. 2, the shock wave front in the x representation is smoothed out in the y representation.

The accuracy and reliability of the present approach is tested at $\mathrm{Re}=10$, and $\mathrm{Re}=100$, for which exact solutions are available. In Table I, the maximum absolute error, L_∞, at four different times for each Re is tabulated and compared with the accurate results obtained using generalized boundary element method by Kakuda and Tosaka (T-K).[6] For $\mathrm{Re}=10$, we choose $A=0.01$ and use only 10 evenly spaced grid points in the y representation. This is to be compared with 40 elements and up to 5 iterations used by the generalized boundary element method. The time increment in both methods is 0.025. The present results are more accurate than those of T-K, while requiring a small number of grid points. For $\mathrm{Re}=100$, the system shows a boundary layer behavior at a sufficiently long time. Accordingly, we choose a large mapping parameter A ($A=0.75$) to concentrate more grid points in the boundary layer region. As a consequence, only 25 grid points are required to obtain solutions which are about five to 200 times more accurate than those obtained using the generalized boundary element method which required 100 elements and up to 6 iterations. The time increment in both methods is 0.01.

To solve Burgers' equation for very large Reynolds number, e.g., $\mathrm{Re}=10^5$, is a difficult task. Kakuda and Tosaka recently reported significantly different results for earlier times from those of Varoglu and Finn,[3] and Nguyen and Reynen.[4] It is not practical to evaluate the analytical solution at this Reynold number due to slow convergence of the infinite series,[14] and thus the exact solution in this regime is unknown. We use a very large mapping parameter ($A=1.560\,68$) to shift most of the 200 grid points into the boundary layer region to obtain oscillation-free solutions. A

TABLE I. Maximum absolute errors of the numerical solutions for Burgers' equation.

	t=0.05		t=0.25		t=0.75		t=1.50	
Re=10	K-T	Present	K-T	Present	K-T	Present	K-T	Present
	3.98(-3)	6.14(-4)	9.03(-3)	7.63(-4)	3.91(-3)	1.66(-4)	1.25(-3)	7.70(-5)
	t=0.40		t=0.80		t=1.20		t=3.00	
Re=100	K-T	Present	K-T	Present	K-T	Present	K-T	Present
	2.60(-2)	3.22(-3)	2.88(-2)	5.98(-3)	1.77(-2)	1.29(-3)	6.93(-3)	2.57(-5)

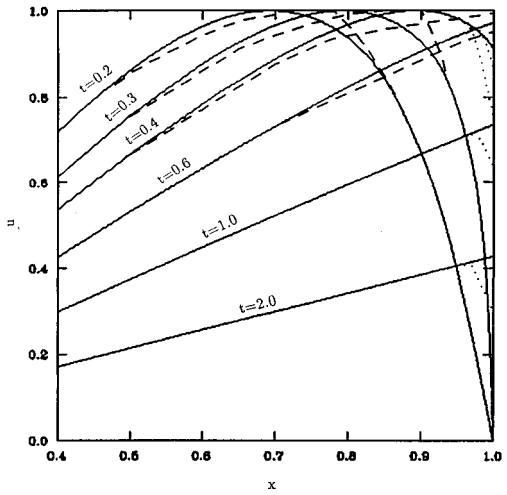

FIG. 3. The solutions of the Burger's equation for $\mathrm{Re}=10^5$. Solid lines: present (200 grid points, second order approximation); Dash lines: Kakuda and Tosaka (200 elements 6 iterations); Dots: Varoglu and Finn (200 elements 5 iterations).

small time increment (2.5×10^{-5}) is chosen to ensure high accuracy. The convergence of the present results is also confirmed by repeating the calculation with 300 grid points. As shown in Fig. 3, at the earlier times ($t=0.2$, 0.4), the present results are close to those of Varoglu and Finn, and of Nguyen and Reynen, but are significantly different from those recently reported by Kakuda and Tosaka. At the later times ($t=1.0-2.0$), the present results are more similar to those obtained by Kakuda and Tosaka than to those reported by Varoglu and Finn at the extreme edge of the boundary layer.

In conclusion, the present DAF-based approach provides a simple and accurate method for solving Burgers' equation for wide range of Reynolds numbers. It is believed that the present approach will also prove useful for solving more general problems in fluid dynamics. Its application to the Navier–Stokes equation is under consideration.

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